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AN APPROXIMATE THEORY FOR HIGH-FREQUENCY VIBRATIONS OF ELASTIC PLATES†

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Abstract—Two-dimensional equations of successively higher orders of approximation for elastic, isotropic plates are deduced from the three-dimensional theory of elasticity by a series expansion in terms of simple thickness-modes for infinite plates. For each order of approximation from the zeroth up to the fourth, kinetic and strain energy densities, stress–strain relations and displacement equations of motion for both flexural and extensional vibrations are presented. Dispersion curves for real and imaginary as well as complex wave numbers in an infinite plate are explored in detail and compared with the solution of the Rayleigh–Lamb frequency equation from the three-dimensional theory.

1. INTRODUCTION

A GENERAL procedure for deducing approximate two-dimensional equations for elastic plates from the three-dimensional theory of elasticity was introduced by Mindlin [1] based on the series-expansion methods of Poisson [2] and Cauchy [3], and the integral method of Kirchhoff [4]. In the same paper, the procedure was applied to a power series expansion and the approximate two-dimensional equations of orders zero and one were obtained. It has been shown that the classical theory of extensional vibrations of thin plates [5] is equivalent to Mindlin's zero order equations. The plate equations for flexural vibrations by Mindlin [6] and Uflyand [7], the equilibrium equations by Reissner [8, 9], two-dimensional analogues of the equations of flexural vibrations for beams by Bresse [10], Timoshenko [11, 12], and Kane and Mindlin's [13] extensional equations for high frequency are either equivalent to or contained in the first order equations of Mindlin. However, in approximations of second and higher orders, inertia terms corresponding to various modes are coupled in the equations of motion due to the lack of orthogonality among the terms of the power series. Using an expansion in series of Legendre polynomials, Mindlin and Medick [14] obtained the equations of extensional vibrations of the second order approximation, for which the coupling of inertia terms is eliminated. As a result of more complicated formulae for the derivatives, when terms higher than the second order are included, complex mathematical forms are still encountered in approximations of the third or higher orders.

In the present paper, using an expansion in a series of simple thickness-modes for infinite plates, two-dimensional equations are deduced from the three-dimensional theory of elasticity by Mindlin's general procedure. Because of the orthogonality of simple thickness-modes and the simple form for the derivatives, approximations of successively higher orders can be obtained with no increase in complication. For the approximate equations, theorems of uniqueness and orthogonality are established.

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Since the simple thickness-modes are the exact limits of the solutions of the three-dimensional theory as wave lengths approach infinity, the cut-off frequencies at zero wave number from the approximate equations are always "exact", and hence the matching of the dispersion curves of the approximate and three-dimensional theories is expected to be very close for long wave lengths and for all frequencies. The results turn out to be as expected except for the lowest flexural and the lowest extensional branches. To improve the matching for these two branches, two correction coefficients, α_1 and α_2, are introduced into the strain and kinetic energy densities for plates.

To extract equations of various orders of approximation from the infinite set, two procedures for series truncation are used. For the zero order approximation, the truncation procedure of Poisson [2] and Mindlin [1] is employed. For any higher Nth order approximation ($N > 0$), a different procedure for series truncation is used for simplicity. For each and every order of approximation from the zeroth up to the fourth order, the energy densities, stress–strain relations and displacement equations of motion are presented. Then the dispersion relations for either flexural motion or extensional motion for an infinite plate are obtained from the approximate equations, and the dispersion curves for real and imaginary as well as complex wave numbers are explored in detail and compared with those from the Rayleigh–Lamb [15, 16] frequency equation of the three-dimensional theory. The close agreement of the results indicates that the applicable range of frequencies for each Nth order theory is $0 \leq \Omega \leq N+1/2$, where the dimensionless frequency $\Omega = \omega/(\pi v_2/2b)$. By studying face-shear vibrations of an infinite plate, it is found that the approximate equations always yield the "exact" dispersion relations.

A method for generating the dispersion relation of any higher order theory is described and applied to the fifth and sixth order approximations.

2. EXPANSION IN SERIES OF THICKNESS-MODES

Consider an infinite plate bounded by a pair of parallel planes $x_2 = \pm b$ as referred to a rectangular coordinate system $x_j (j = 1, 2, 3)$. Series of vibrational modes which are independent of x_1, x_3 coordinates and correspond to traction-free faces were discussed in detail by Mindlin [1] and are called simple thickness-modes. The displacement components corresponding to these modes may be written as

$$u_j = A_{jn} \cos \frac{n\pi}{2}\left(1 - \frac{x_2}{b}\right) e^{i\omega_n t} \tag{1}$$

where $n = 1, 2, 3, \ldots$, $\omega_n = n\pi v_1/2b$ are the resonant frequencies of the infinite plate vibrating in thickness-stretch modes, while $\omega_n = n\pi v_2/2b$ are the resonant frequencies of thickness-shear modes and $v_1 = \sqrt{[(\lambda+2\mu)/\rho]}$ and $v_2 = \sqrt{(\mu/\rho)}$ are the dilatational and equivoluminal wave velocities, λ, μ being Lamé constants.

By expanding the general displacement components in an infinite series with their x_2-dependence expressed by the simple thickness-modes, one may write

$$u_j(x_1, x_2, x_3, t) = \sum_{n=0}^{\infty} \cos \frac{n\pi}{2}(1-\eta) u_j^{(n)}(x_1, x_3, t) \tag{2}$$

where $\eta = x_2/b$, $u_j(x_1, x_3, t)$ are functions of x_1, x_3 and t only, and are called the *nth order displacements* although they represent the amplitudes of the nth thickness-mode distribution of displacements across the thickness of the plate. The distributions of displacements across the thickness associated with $u_j^{(n)}$ for $n = 0, 1, 2$ are illustrated in Fig. 1.

Stress equations of motion

From the variational equation of motion [17], one obtains

$$\int_V (\tau_{ij,i} - \rho u_{j,tt})\delta u_j \, dV = 0 \tag{3}$$

where dV, the volume element, can be replaced by $bd\eta \, dA$ with $dA = dx_1 \, dx_3$ representing the area element of the plate. By substitution of (2) into (3), integration with respect to η

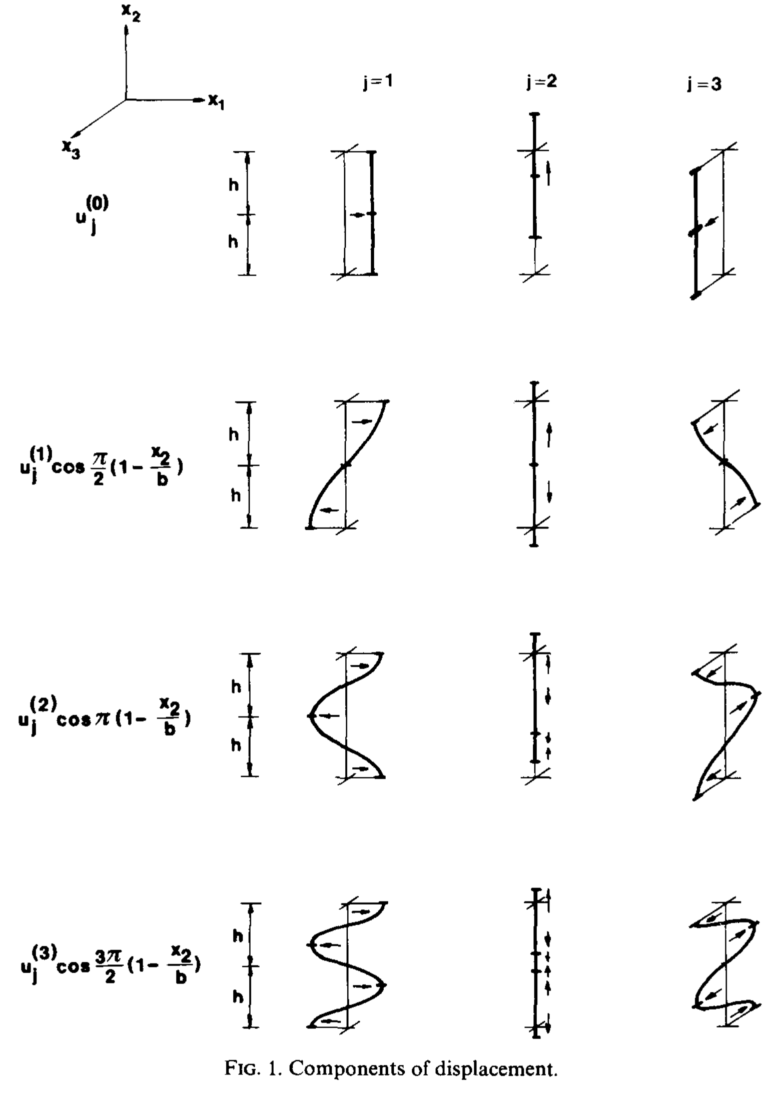


FIG. 1. Components of displacement.

over the interval $[-1, 1]$ and use of the following identities:

$$\int_{-1}^{1} \sin\frac{m\pi}{2}(1-\eta)\sin\frac{n\pi}{2}(1-\eta)\,d\eta = \delta_{mn}$$
$$\int_{-1}^{1} \cos\frac{m\pi}{2}(1-\eta)\cos\frac{n\pi}{2}(1-\eta)\,d\eta = \delta_{mn} \tag{4}$$

and

$$\int_{-1}^{1} \sin\frac{m\pi}{2}(1-\eta)\cos\frac{n\pi}{2}(1-\eta)\,d\eta = A_{mn} = \begin{cases} 0, & m+n \text{ even} \\ \dfrac{4m}{(m^2-n^2)\pi}, & m+n \text{ odd} \end{cases}$$

one obtains

$$\int_A \sum_{n=0}^{\infty} \left(\tau_{ij,i}^{(n)} - \frac{n\pi}{2b}\bar{\tau}_{2j}^{(n)} + \frac{1}{b}F_j^{(n)} - \rho u_{j,tt}^{(n)}\right)\delta u_j^{(n)}\,dA = 0 \tag{5}$$

where

$$\tau_{ij}^{(n)} \equiv \int_{-1}^{1} \tau_{ij}\cos\frac{n\pi}{2}(1-\eta)\,d\eta, \qquad \bar{\tau}_{ij}^{(n)} \equiv \int_{-1}^{1} \tau_{ij}\sin\frac{n\pi}{2}(1-\eta)\,d\eta,$$
$$F_j^{(n)} \equiv \left[\tau_{2j}\cos\frac{n\pi}{2}(1-\eta)\right]_{-1}^{1} = \tau_{2j}(b) - (-1)^n\tau_{2j}(-b). \tag{6}$$

It may be noted that the quantities defined above are functions of x_1, x_3 and t only. $\tau_{ij}^{(n)}$ and $\bar{\tau}_{ij}^{(n)}$ are called *nth order components of stress* but with $\cos(n\pi/2)(1-\eta)$ and $\sin(2\pi/2)(1-\eta)$ as weighting functions respectively, while $F_j^{(n)}$ are called the *nth order components of face-traction*. Since (5) must hold for arbitrary A and for every arbitrary $\delta u_j^{(n)}$ and if in addition it is assumed that the integrand of (5) is continuous, then the quantity in the parentheses must be zero. Hence the *nth order stress equations of motion are*

$$\tau_{ij,i}^{(n)} - \frac{n\pi}{2b}\bar{\tau}_{2j}^{(n)} + \frac{1}{b}F_j^{(n)} = \rho u_{j,tt}^{(n)}. \tag{7}$$

Equations (7) are similar in form to Mindlin's approximate plate equations. In the equation of motion obtained by Mindlin [1] by using an expansion in a power series, an infinite series appears on the right hand side and no series appears on the left hand side; while in the analogous equations obtained by Mindlin and Medick [14], using an expansion in a series of Legendre polynomials, a finite series of terms depending on the order n appears instead of the second term on the left hand side of (7), and no series appears on the right hand side. Hence (7) has a simpler form than the aforementioned equations, especially when n becomes large.

Strains and stress–strain relations

Inserting (2) into the strains in the three-dimensional theory

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{j,i} + u_{i,j}), \tag{8}$$

one obtains

$$\varepsilon_{ij} = \sum_{n=0}^{\infty}\left[\varepsilon_{ij}^{(n)}\cos\frac{n\pi}{2}(1-\eta) + \bar{\varepsilon}_{ij}^{(n)}\sin\frac{n\pi}{2}(1-\eta)\right], \tag{9}$$

where

$$\varepsilon_{ij}^{(n)} \equiv \tfrac{1}{2}(u_{j,i}^{(n)} + u_{i,j}^{(n)}), \tag{10}$$

$$\bar{\varepsilon}_{ij}^{(n)} \equiv \frac{n\pi}{4b}(\delta_{2i}u_j^{(n)} + \delta_{2j}u_i^{(n)}), \tag{11}$$

and they are called the n*th order components of strain.*

In the three-dimensional theory, the constitutive relations for elastic and isotropic materials are

$$\tau_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}, \tag{12}$$

where λ, μ are the Lamé constants. Substitution of (9) into (12), then, in turn, into the first two equations of (6) yields, respectively, the following n*th order stress–strain relations*

$$\tau_{ij}^{(n)} = \lambda\delta_{ij}\varepsilon_{kk}^{(n)} + 2\mu\varepsilon_{ij}^{(n)} + \sum_{m=0}^{\infty} A_{mn}(\lambda\delta_{ij}\bar{\varepsilon}_{kk}^{(m)} + 2\mu\bar{\varepsilon}_{ij}^{(m)}) \tag{13}$$

$$\bar{\tau}_{ij}^{(n)} = \lambda\delta_{ij}\bar{\varepsilon}_{kk}^{(n)} + 2\mu\bar{\varepsilon}_{ij}^{(n)} + \sum_{m=0}^{\infty} A_{nm}(\lambda\delta_{ij}\varepsilon_{kk}^{(m)} + 2\mu\varepsilon_{ij}^{(m)}). \tag{14}$$

It may be seen that two sets of stresses and strains are introduced in the present paper and that they are denoted by $\tau_{ij}^{(n)}$, $\varepsilon_{ij}^{(n)}$ and $\bar{\tau}_{ij}^{(n)}$, $\bar{\varepsilon}_{ij}^{(n)}$, respectively. The stress components $\tau_{ij}^{(n)}$, as can be seen by the definition, are similar to Mindlin's nth order components of stress [18]. Furthermore, if (7) are compared with the analogous equations by Mindlin [18], one may note that $\bar{\tau}_{ij}^{(n)}$ play similar roles to the $\tau_{ij}^{(n-1)}$ of Mindlin's theory. The strain components defined in (10) have the same form as the strains in the three-dimensional theory. However the $\varepsilon_{22}^{(n)}$ are identically equal to zero. An inspection of (11) reveals that $\bar{\varepsilon}_{ij}^{(n)} = 0$ for $i, j = 1, 3$ and the three non-zero components $\bar{\varepsilon}_{2j}^{(n)}$ are proportional to $u_j^{(n)}$ for $j = 1, 2, 3$. The strain components $\varepsilon_{ij}^{(n)}$ and $\bar{\varepsilon}_{22}^{(n+1)}$ for $n = 0, 1, 2$ are illustrated in Fig. 2, in which the $\bar{\varepsilon}_{22}^{(n+1)}$ take the place of the vanishing $\varepsilon_{22}^{(n)}$.

Energy densities

Let $U = \tau_{ij}\varepsilon_{ij}/2$ be the strain energy density in the three-dimensional theory; the plate-strain energy density is then defined as

$$\bar{U} \equiv \int_{-1}^{1} U\, d\eta. \tag{15}$$

By inserting (9) into above and using (10) and (11), one finds

$$2\bar{U} = \sum_{n=0}^{\infty} (\tau_{ij}^{(n)}\varepsilon_{ij}^{(n)} + \bar{\tau}_{ij}^{(n)}\bar{\varepsilon}_{ij}^{(n)}), \tag{16}$$

or

$$2\bar{U} = \sum_{n=0}^{\infty}\left[\lambda\delta_{ij}\varepsilon_{kk}^{(n)} + 2\mu\varepsilon_{ij}^{(n)} + \sum_{m=0}^{\infty} A_{mn}(\lambda\delta_{ij}\bar{\varepsilon}_{kk}^{(m)} + 2\mu\bar{\varepsilon}_{ij}^{(m)})\right]\varepsilon_{ij}^{(n)}$$
$$+ \sum_{n=0}^{\infty}\left[\lambda\delta_{ij}\bar{\varepsilon}_{kk}^{(n)} + 2\mu\bar{\varepsilon}_{ij}^{(n)} + \sum_{m=0}^{\infty} A_{nm}(\lambda\delta_{ij}\varepsilon_{kk}^{(m)} + 2\mu\varepsilon_{ij}^{(m)})\right]\bar{\varepsilon}_{ij}^{(n)}. \tag{17}$$

One may note that

$$\tau_{ij}^{(n)} = \frac{\partial\bar{U}}{\partial\varepsilon_{ij}^{(n)}}, \qquad \bar{\tau}_{ij}^{(n)} = \frac{\partial\bar{U}}{\partial\bar{\varepsilon}_{ij}^{(n)}}. \tag{18}$$

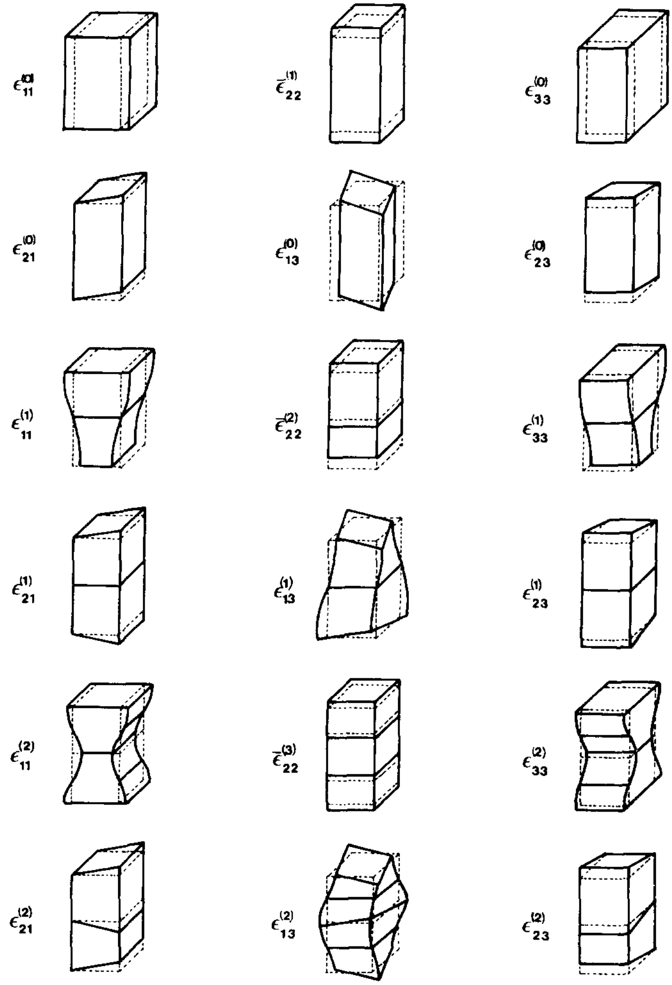


FIG. 2. Components of strain.

Similarly, the plate-kinetic energy density is defined by

$$\bar{K} \equiv \int_{-1}^{1} K \,\mathrm{d}\eta = \frac{1}{2}\rho \sum_{n=0}^{\infty} u_{i,t}^{(n)} u_{i,t}^{(n)} \tag{19}$$

where $K = \rho u_{i,t} u_{i,t}/2$ is the kinetic energy density in the three-dimensional theory.

3. UNIQUENESS AND ORTHOGONALITY

A theorem analogous to Neumann's [19] for the uniqueness of solutions of the approximate equations may be obtained. Form the following expression from (7):

$$\int_{t_0}^{t_1} \mathrm{d}t \int_A \sum_{n=0}^{\infty} \left(\tau_{ij,i}^{(n)} - \frac{n\pi}{2b}\bar{\tau}_{2j}^{(n)} + F_j^{(n)}/b - \rho u_{j,tt}^{(n)}\right) u_{j,t}^{(n)} \,\mathrm{d}A = 0.$$

Use the two-dimensional divergence theorem, (16) and (19), and note that in (17)

$$\tau_{ij}^{(n)}\varepsilon_{ij}^{(n)} = \tau_{ij}^{(n)}u_{i,j}^{(n)} \qquad \text{and} \qquad \bar{\tau}_{ij}^{(n)}\bar{\varepsilon}_{ij}^{(n)} = \frac{n\pi}{2b}\bar{\tau}_{2j}^{(n)}u_j^{(n)}.$$

An energy equation results:

$$\int_A [\bar{K}(t_1)+\bar{U}(t_1)]\,\mathrm{d}A = \int_A [\bar{K}(t_0)+\bar{U}(t_0)]\,\mathrm{d}A$$
$$+\int_{t_0}^{t_1}\mathrm{d}t\oint_C \sum_{n=0}^{\infty} \nu_i\tau_{ij}^{(n)}u_{j,t}^{(n)}\,\mathrm{d}S+\int_{t_0}^{t_1}\mathrm{d}t\int_A b^{-1}\sum_{n=0}^{\infty} F_j^{(n)}u_{j,t}^{(n)}\,\mathrm{d}A, \tag{20}$$

where C is the edge around the plate and ν_i are the components of the unit outward normal to C in the plane of the plate. By the usual argument based upon the positive definiteness of $\bar{K}$ and $\bar{U}$, the sufficient conditions for a unique solution are obtained in the absence of discontinuities and singularities:

1. Initial values of $u_j^{(n)}$ and $u_{j,t}^{(n)}$ for each and every order of n at each point of the plate.
2. One member of each of the products of $F_1^{(n)}u_1^{(n)}$, $F_2^{(n)}u_2^{(n)}$ and $F_3^{(n)}u_3^{(n)}$ for each and every order of n and at each point of the plate.
3. One member of each of the products $\tau_{rr}^{(n)}u_r^{(n)}$, $\tau_{rs}^{(n)}u_s^{(n)}$ and $\tau_{r2}^{(n)}u_2^{(n)}$ for each and every order of n and at each point on the edge C, where r, s are referred to as the normal and tangential directions to the edge C.

It may be seen that the stress components $\bar{\tau}_{ij}^{(n)}$ do not appear in any one of the above stated conditions.

Consider two sets of solutions

$$u_j^{(n)} = u_j^{(n)a}(x_1, x_3)\,\mathrm{e}^{i\omega_a t}$$
$$u_j^{(n)} = u_j^{(n)b}(x_1, x_3)\,\mathrm{e}^{i\omega_b t},$$

which satisfy the homogeneous equations

$$\tau_{ij,i}^{(n)a}-\frac{n\pi}{2b}\bar{\tau}_{2j}^{(n)a} = -\rho\omega_a^2 u_j^{(n)a} \tag{21}$$

and

$$\tau_{ij,i}^{(n)b}-\frac{n\pi}{2b}\bar{\tau}_{ij}^{(n)b} = -\rho\omega_b^2 u_j^{(n)b}, \tag{22}$$

respectively. A theorem concerning the orthogonality between functions $u_j^{(n)a}$ and $u_j^{(n)b}$ may be established in a manner similar to that of the Clebsch theorem [20]. Multiplying (21) by $+u_j^{(n)b}$ and (22) by $-u_j^{(n)a}$, respectively for every n, summing and integrating over the area of the plate and, in the resulting expression, replacing $\tau_{ij}^{(n)}u_{j,i}^{(n)}$ terms by $(\tau_{ij}^{(n)}u_j^{(n)})_{,i}-\tau_{ij,i}^{(n)}u_j^{(n)}$ and finally applying the divergence theorem, one obtains

$$\oint_C \sum_{n=0}^{\infty} \nu_i(\tau_{ij}^{(n)a}u_j^{(n)b}-\tau_{ij}^{(n)b}u_j^{(n)a})\,\mathrm{d}s-\int_A \sum_{n=0}^{\infty} [(\tau_{ij}^{(n)a}\varepsilon_{ij}^{(n)b}-\tau_{ij}^{(n)b}\varepsilon_{ij}^{(n)a})$$
$$+(\bar{\tau}_{ij}^{(n)a}\bar{\varepsilon}_{ij}^{(n)b}-\bar{\tau}_{ij}^{(n)b}\bar{\varepsilon}_{ij}^{(n)a})]\,\mathrm{d}A = -\rho(\omega_a^2-\omega_b^2)\int_A \sum_{n=0}^{\infty} u_j^{(n)a}u_j^{(n)b}\,\mathrm{d}A.$$

On the left hand side of the above equation, the first term vanishes for homogeneous conditions on the edge and the second term vanishes identically, as can be verified by the direct substitution of (13), (14). Therefore, if $\omega_a \neq \omega_b$,

$$\int_A \sum_{n=0}^{\infty} u_j^{(n)a} u_j^{(n)b} \, \mathrm{d}A = 0. \tag{23}$$

4. TRUNCATION PROCEDURES

By the series expansion of the displacement, as described in Section 2, the three-dimensional equations of elasticity, i.e. (3), (8) and (12), are replaced by an infinite set of two-dimensional plate equations given by (6), (7), (11), (12) and (13), (14). To extract approximate theories of various orders from this infinite set of equations, truncation procedures for the series are described below.

For the zero order approximation

The same procedure used by Poisson [2] and Mindlin [1] for zero order approximation is employed here by setting

$$\begin{aligned} &\text{(a) } u_1^{(n)} = 0,\ u_3^{(n)} = 0,\ n > 0 \\ &\text{(b) } u_2^{(n)} = 0,\ n > 1 \\ &\text{(c) } \tau_{22}^{(0)} = 0 \text{ and } u_{2,t}^{(1)} = 0. \end{aligned} \tag{24}$$

For the N*th order approximation* ($N > 0$)

The truncation procedure for the Nth order theory with N any positive integer is to set

$$u_j^{(n)} = 0, \qquad n > N \tag{25}$$

and to ignore the stress components $\tau_{ij}^{(n)}$ and $\bar{\tau}_{ij}^{(n)}$ for $n > N$.

5. CORRECTION COEFFICIENTS α_1 AND α_2

By comparing the dispersion curves, for straight-crested waves propagating in the x_1 direction, from the approximate plate theory of order N (which is obtained by the procedure described in the preceding section and will be presented in detail in Section 6), with those from the Rayleigh–Lamb frequency equation [15, 16] of the three-dimensional theory, it is found that the approximate theory always yields the "exact" cut-off frequencies and two sets of dispersion curves match quite well for frequencies up to $\Omega = N + 1/2$ and wave numbers $|z| \leq N + 1$, where the dimensionless frequency and wave number are defined by

$$\Omega = \omega \bigg/ \left(\frac{\pi v_2}{2b}\right), \qquad z = \xi \bigg/ \left(\frac{\pi}{2b}\right) \tag{26}$$

except for the lowest flexural and the lowest extensional branches.

To adjust the energy densities, given in (17) and (19), for better matching of the lowest flexural and extensional branches, two correction factors α_1 and α_2 are introduced as below.

$$\begin{aligned} 2\bar{U} = {} & \lambda(\varepsilon_{ii}^{(0)}\varepsilon_{jj}^{(0)} + \varepsilon_{ii}^{(1)}\varepsilon_{jj}^{(1)} + \dots) + 2\mu(\varepsilon_{ij}^{(0)}\varepsilon_{ij}^{(0)} + \alpha_2^p \varepsilon_{ij}^{(1)}\varepsilon_{ij}^{(1)} + \dots) \\ & + \varepsilon_{ij}^{(0)}[A_{10}\alpha_1(\lambda\delta_{ij}\bar{\varepsilon}_{kk}^{(1)} + 2\mu\bar{\varepsilon}_{ij}^{(1)}) + A_{30}(\lambda\delta_{ij}\bar{\varepsilon}_{kk}^{(3)} + 2\mu\bar{\varepsilon}_{ij}^{(3)} + \dots] \\ & + \varepsilon_{ij}^{(1)}[A_{21}(\lambda\delta_{ij}\bar{\varepsilon}_{kk}^{(2)} + 2\mu\bar{\varepsilon}_{ij}^{(2)}) + A_{41}(\lambda\delta_{ij}\bar{\varepsilon}_{kk}^{(4)} + 2\mu\bar{\varepsilon}_{ij}^{(4)}) + \dots] \\ & + \varepsilon_{ij}^{(2)}[A_{12}(\lambda\delta_{ij}\bar{\varepsilon}_{kk}^{(1)} + 2\mu\bar{\varepsilon}_{ij}^{(1)}) + A_{32}(\lambda\delta_{ij}\bar{\varepsilon}_{kk}^{(3)} + 2\mu\bar{\varepsilon}_{ij}^{(3)} + \dots] \\ & + \dots \end{aligned}$$

$$
\begin{aligned}
&+\lambda(\bar{\varepsilon}_{ii}^{(0)}\bar{\varepsilon}_{jj}^{(0)}+\bar{\varepsilon}_{ii}^{(1)}\bar{\varepsilon}_{jj}^{(1)}+\ldots)+2\mu(\bar{\varepsilon}_{ij}^{(0)}\bar{\varepsilon}_{ij}^{(0)}+\bar{\varepsilon}_{ij}^{(1)}\bar{\varepsilon}_{ij}^{(1)}+\ldots)\\
&+\bar{\varepsilon}_{ij}^{(1)}[A_{10}\alpha_1(\lambda\delta_{ij}\varepsilon_{ij}^{(0)}+2\mu\varepsilon_{ij}^{(0)})+A_{12}(\lambda\delta_{ij}\varepsilon_{kk}^{(2)}+2\mu\varepsilon_{ij}^{(2)})+\ldots]\\
&+\bar{\varepsilon}_{ij}^{(2)}[A_{21}(\lambda\delta_{ij}\varepsilon_{kk}^{(1)}+2\mu\varepsilon_{ij}^{(1)})+A_{23}(\lambda\delta_{ij}\varepsilon_{kk}^{(3)}+2\mu\varepsilon_{ij}^{(3)})+\ldots]\\
&+\ldots
\end{aligned}
\tag{27}
$$

and

$$2\bar{K} = \rho(\alpha_2^{-p}u_{i,t}^{(0)}u_{i,t}^{(0)}+u_{i,t}^{(1)}u_{i,t}^{(1)}+u_{i,t}^{(2)}u_{i,t}^{(2)}+\ldots), \tag{28}$$

where

$$p = \cos^2(i\pi/2).$$

It can be seen that α_1 is introduced into $\bar{U}$ for the strains $\varepsilon_{ij}^{(0)}$ and $\bar{\varepsilon}_{ij}^{(1)}$ which are associated with the coefficient A_{10}, while α_2 is introduced for the $\varepsilon_{2j}^{(1)}\varepsilon_{2j}^{(1)}$ term in $\bar{U}$ and $u_{2,t}^{(0)}u_{2,t}^{(0)}$ term in $\bar{K}$.

It will be shown in the next section that in order to make the slope of the lowest flexural branch in the first order theory and the slope of the lowest extensional branch in the second order theory coincide, respectively, with those from the three-dimensional theory when both the frequency and wave number approach zero, the value of α_1 must be taken as

$$\alpha_1 = \pi/4. \tag{29}$$

In order to make the phase velocities of the lowest flexural and the lowest extensional branches approach that of the Rayleigh surface waves [21] as both the values of the frequency and wave number become large, α_2 must be set equal to the real root of the following equation [6]

$$\alpha_2^3-8\alpha_2^2+8(3-2/k^2)\alpha_2-16(1-1/k^2) = 0$$

for Rayleigh surface waves, where $k^2 = v_1^2/v_2^2 = 2(1-\nu)/(1-2\nu)$. The values of α_2 for different values of Poisson's ratio are given below.

TABLE 1

ν	0·00	0·100	0·200	0·250	0·300	0·350	0·400	0·500
α_2	0·764	0·798	0·830	0·845	0·860	0·874	0·888	0·963

The adjusted energy densities $\bar{U}$ and $\bar{K}$ are still positive definite if, in addition to the usual requirements, $3\lambda+2\mu > 0$, $\mu > 0$, one requires that $\alpha_2 > 0$. After the energy densities are adjusted, the strain–displacement equations remain the same. The stress equations of motion derived from $\bar{U}$ and $\bar{K}$ by the variational principle also remain unchanged except that the inertia term $\rho u_{2,tt}^{(0)}$ is replaced by $\rho u_{2,tt}^{(0)}/\alpha_2$. The stress components derived from the adjusted $\bar{U}$ by (18) for the first five orders are given as follows.

$$
\begin{aligned}
\tau_{ij}^{(0)} &= \sigma_{ij}^{(0)}+\frac{4}{\pi}\alpha_1\bar{\sigma}_{ij}^{(1)}+\frac{4}{3\pi}\bar{\sigma}_{ij}^{(3)}+\frac{4}{5\pi}\bar{\sigma}_{ij}^{(5)}+\ldots\\
\tau_{ij}^{(1)} &= \alpha_2^p\sigma_{ij}^{(1)}+\frac{8}{3\pi}\bar{\sigma}_{ij}^{(2)}+\frac{16}{15\pi}\bar{\sigma}_{ij}^{(4)}+\ldots\\
\tau_{ij}^{(2)} &= \sigma_{ij}^{(2)}-\frac{4}{3\pi}\bar{\sigma}_{ij}^{(1)}+\frac{12}{5\pi}\bar{\sigma}_{ij}^{(3)}+\frac{20}{21\pi}\bar{\sigma}_{ij}^{(5)}+\ldots\\
\tau_{ij}^{(3)} &= \sigma_{ij}^{(3)}-\frac{8}{5\pi}\bar{\sigma}_{ij}^{(2)}+\frac{16}{7\pi}\bar{\sigma}_{ij}^{(4)}+\ldots
\end{aligned}
\tag{30}
$$

$$\tau_{ij}^{(4)} = \sigma_{ij}^{(4)} - \frac{4}{15\pi}\bar{\sigma}_{ij}^{(1)} - \frac{12}{7\pi}\bar{\sigma}_{ij}^{(3)} + \frac{20}{9\pi}\bar{\sigma}_{ij}^{(5)} + \dots$$

$$\tau_{ij}^{(5)} = \sigma_{ij}^{(5)} - \frac{8}{21\pi}\bar{\sigma}_{ij}^{(2)} - \frac{16}{9\pi}\bar{\sigma}_{ij}^{(4)} + \dots \qquad \text{(30 continued)}$$

and

$$\bar{\tau}_{ij}^{(0)} = 0$$

$$\bar{\tau}_{ij}^{(1)} = \bar{\sigma}_{ij}^{(1)} + \frac{4}{\pi}\alpha_1\sigma_{ij}^{(0)} - \frac{4}{3\pi}\sigma_{ij}^{(2)} - \frac{4}{15\pi}\sigma_{ij}^{(4)} + \dots$$

$$\bar{\tau}_{ij}^{(2)} = \bar{\sigma}_{ij}^{(2)} + \frac{8}{3\pi}\sigma_{ij}^{(1)} - \frac{8}{5\pi}\sigma_{ij}^{(3)} - \frac{8}{21\pi}\sigma_{ij}^{(5)} + \dots$$

$$\bar{\tau}_{ij}^{(3)} = \bar{\sigma}_{ij}^{(3)} + \frac{12}{9\pi}\sigma_{ij}^{(0)} + \frac{12}{5\pi}\sigma_{ij}^{(2)} - \frac{12}{7\pi}\sigma_{ij}^{(4)} + \dots \qquad (31)$$

$$\bar{\tau}_{ij}^{(4)} = \bar{\sigma}_{ij}^{(4)} + \frac{16}{15\pi}\sigma_{ij}^{(1)} + \frac{16}{7\pi}\sigma_{ij}^{(3)} - \frac{16}{9\pi}\sigma_{ij}^{(5)} + \dots$$

$$\bar{\tau}_{ij}^{(5)} = \bar{\sigma}_{ij}^{(5)} + \frac{20}{25\pi}\sigma_{ij}^{(0)} + \frac{20}{21\pi}\sigma_{ij}^{(2)} + \frac{20}{9\pi}\sigma_{ij}^{(4)} + \dots$$

where

$$\sigma_{ij}^{(n)} \equiv \lambda\delta_{ij}\varepsilon_{kk}^{(n)} + 2\mu\varepsilon_{ij}^{(n)}, \qquad \bar{\sigma}_{ij}^{(n)} \equiv \lambda\delta_{ij}\bar{\varepsilon}_{kk}^{(n)} + 2\mu\bar{\varepsilon}_{ij}^{(n)}. \qquad (32)$$

The stress–displacement relations may be obtained by inserting (10) and (11) into (32) and then, in turn, into (30) and (31). For ease of reference and in order to avoid repetition later the stress–displacement relations for the first five orders are listed below, in which the subscript a takes on only the values 1 and 3 and

$$\mathrm{e}^{(n)} \equiv u_{1,1}^{(n)} + u_{3,3}^{(n)}. \qquad (33)$$

Zero order

$$\tau_{aa}^{(0)} = \lambda\,\mathrm{e}^{(0)} + 2\mu u_{a,a}^{(0)} + \frac{2}{b}\lambda(u_1^{(1)} + u_2^{(3)} + u_2^{(5)} + \dots), \text{ no sum}$$

$$\tau_{22}^{(0)} = \lambda\,\mathrm{e}^{(0)} + \frac{2}{b}(\lambda+2\mu)(\alpha_1 u_2^{(1)} + u_2^{(3)} + u_2^{(5)} + \dots),$$

$$\tau_{2a}^{(0)} = \mu u_{2,a}^{(0)} + \frac{2}{b}\mu(\alpha_1 u_a^{(1)} + u_a^{(3)} + u_a^{(5)} + \dots), \qquad (34)$$

$$\tau_{13}^{(0)} = \mu(u_{3,1}^{(0)} + u_{1,3}^{(0)}).$$

First order

$$\tau_{aa}^{(1)} = \lambda\,\mathrm{e}^{(1)} + 2\mu u_{a,a}^{(1)} + \frac{8}{b}\lambda\left(\frac{1}{3}u_2^{(2)} + \frac{4}{15}u_2^{(4)} + \dots\right), \text{ no sum}$$

$$\tau_{22}^{(1)} = \alpha_2\lambda\,\mathrm{e}^{(1)} + \frac{8}{b}(\lambda+2\mu)\left(\frac{1}{3}u_2^{(2)} + \frac{4}{15}u_2^{(4)} + \dots\right),$$

$$\tau_{2a}^{(1)} = \alpha_2\lambda u_{2,a}^{(1)} + \frac{8}{b}\mu\left(\frac{1}{3}u_a^{(2)} + \frac{4}{15}u_a^{(4)} + \dots\right),$$

$$\tau_{13}^{(1)} = \mu(u_{3,1}^{(1)} + u_{1,3}^{(1)}), \qquad (35)$$

$$\bar{\tau}_{22}^{(1)} = \frac{\pi}{2b}(\lambda+2\mu)u_2^{(1)} + \frac{4}{\pi}\lambda\left(\alpha_1\,\mathrm{e}^{(0)} - \frac{1}{3}\mathrm{e}^{(2)} - \frac{1}{15}\mathrm{e}^{(4)} + \dots\right)$$

$$\bar{\tau}_{2a}^{(1)} = \frac{\pi}{2b}\mu u_a^{(1)} + \frac{4}{\pi}\mu\left(\alpha_1 u_{2,a}^{(0)} - \frac{1}{3}u_{2,a}^{(2)} - \frac{1}{15}u_{2,a}^{(4)} + \dots\right).$$

Second order

$$
\begin{aligned}
\tau_{aa}^{(2)} &= \lambda\,\mathrm{e}^{(2)} + 2\mu u_{a,a}^{(2)} + \frac{2}{b}\lambda\left(-\frac{1}{3}u_2^{(1)} + \frac{9}{5}u_2^{(3)} - \frac{25}{21}u_2^{(5)} + \dots\right), \text{ no sum}\\
\tau_{22}^{(2)} &= \lambda\,\mathrm{e}^{(2)} + \frac{2}{b}(\lambda+2\mu)\left(-\frac{1}{3}u_2^{(1)} + \frac{9}{5}u_2^{(3)} + \frac{25}{21}u_2^{(5)} + \dots\right),\\
\tau_{2a}^{(2)} &= \mu u_{2,a}^{(2)} + \frac{2}{b}\mu\left(-\frac{1}{3}u_a^{(1)} + \frac{9}{5}u_a^{(3)} + \frac{25}{21}u_a^{(5)} + \dots\right),\\
\tau_{13}^{(2)} &= \mu(u_{3,1}^{(2)} + u_{1,3}^{(2)}),\\
\bar{\tau}_{22}^{(2)} &= \frac{\pi}{b}(\lambda+2\mu)u_2^{(2)} + \frac{8}{\pi}\lambda\left(\frac{1}{3}\mathrm{e}^{(1)} - \frac{1}{5}\mathrm{e}^{(3)} - \frac{1}{21}\mathrm{e}^{(5)} + \dots\right),\\
\bar{\tau}_{2a}^{(2)} &= \frac{\pi}{b}\mu u_a^{(2)} + \frac{8}{\pi}\mu\left(\frac{1}{3}u_{2,a}^{(1)} - \frac{1}{5}u_{2,a}^{(3)} - \frac{1}{21}u_{2,a}^{(5)} + \dots\right).
\end{aligned} \tag{36}
$$

Third order

$$
\begin{aligned}
\tau_{aa}^{(3)} &= \lambda\,\mathrm{e}^{(3)} + 2\mu u_{a,a}^{(3)} + \frac{8}{b}\lambda\left(-\frac{1}{5}u_2^{(1)} + \frac{4}{7}u_2^{(3)} + \frac{1}{3}u_2^{(5)} + \dots\right), \text{ no sum}\\
\tau_{22}^{(3)} &= \lambda\,\mathrm{e}^{(3)} + \frac{8}{b}(\lambda+2\mu)\left(-\frac{1}{5}u_2^{(1)} + \frac{4}{7}u_2^{(3)} - \frac{1}{3}u_2^{(5)} + \dots\right),\\
\tau_{2a}^{(3)} &= \mu u_{2,a}^{(3)} + \frac{8}{b}\mu\left(-\frac{1}{5}u_a^{(1)} + \frac{4}{7}u_a^{(3)} + \frac{1}{3}u_a^{(5)} + \dots\right),\\
\tau_{13}^{(3)} &= \mu(u_{3,1}^{(3)} + u_{1,3}^{(3)}),\\
\bar{\tau}_{22}^{(3)} &= \frac{3\pi}{2b}(\lambda+2\mu)u_2^{(3)} + \frac{12}{\pi}\lambda\left(\frac{1}{9}\mathrm{e}^{(0)} + \frac{1}{5}\mathrm{e}^{(2)} - \frac{1}{7}\mathrm{e}^{(4)} + \dots\right),\\
\bar{\tau}_{2a}^{(3)} &= \frac{3\pi}{2b}\mu u_a^{(3)} + \frac{12}{\pi}\mu\left(\frac{1}{9}u_{2,a}^{(0)} + \frac{1}{5}u_{2,a}^{(2)} - \frac{1}{7}u_{2,a}^{(4)} + \dots\right).
\end{aligned} \tag{37}
$$

Fourth order

$$
\begin{aligned}
\tau_{aa}^{(4)} &= \lambda\,\mathrm{e}^{(4)} + 2\mu u_{a,a}^{(4)} + \frac{2}{b}\lambda\left(-\frac{1}{15}u_2^{(1)} - \frac{9}{7}u_2^{(3)} + \frac{25}{9}u_2^{(5)} + \dots\right), \text{ no sum}\\
\tau_{22}^{(4)} &= \lambda\,\mathrm{e}^{(4)} + \frac{2}{b}(\lambda+2\mu)\left(-\frac{1}{15}u_2^{(1)} - \frac{9}{7}u_2^{(3)} + \frac{25}{9}u_2^{(5)} + \dots\right),\\
\tau_{2a}^{(4)} &= \mu u_{2,a}^{(4)} + \frac{2}{b}\mu\left(-\frac{1}{15}u_a^{(1)} - \frac{9}{7}u_a^{(3)} + \frac{25}{9}u_a^{(5)} + \dots\right),\\
\tau_{13}^{(4)} &= \mu(u_{3,1}^{(4)} + u_{1,3}^{(4)}),\\
\bar{\tau}_{22}^{(4)} &= \frac{2\pi}{b}(\lambda+2\mu)u_2^{(4)} + \frac{16}{\pi}\lambda\left(\frac{1}{15}\mathrm{e}^{(1)} + \frac{1}{7}\mathrm{e}^{(3)} - \frac{1}{9}\mathrm{e}^{(5)} + \dots\right),\\
\bar{\tau}_{2a}^{(4)} &= \frac{2\pi}{b}\mu u_a^{(4)} + \frac{16}{\pi}\mu\left(\frac{1}{15}u_{2,a}^{(1)} + \frac{1}{7}u_{2,a}^{(3)} - \frac{1}{9}u_{2,a}^{(5)} + \dots\right).
\end{aligned} \tag{38}
$$

Fifth order

$$
\begin{aligned}
\tau_{aa}^{(5)} &= \lambda\, e^{(5)} + 2\mu u_{a,a}^{(5)} + \frac{8}{b}\lambda\left(-\frac{1}{21}u_2^{(2)} - \frac{4}{9}u_2^{(4)} + \dots\right), \text{ no sum}\\
\tau_{22}^{(5)} &= \lambda\, e^{(5)} + \frac{8}{b}(\lambda+2\mu)\left(-\frac{1}{21}u_2^{(2)} - \frac{4}{9}u_2^{(4)} + \dots\right),\\
\tau_{2a}^{(5)} &= \mu u_{2,a}^{(5)} + \frac{8}{b}\mu\left(-\frac{1}{21}u_a^{(2)} - \frac{4}{9}u_a^{(4)} + \dots\right),\\
\tau_{13}^{(5)} &= \mu(u_{3,1}^{(5)} + u_{1,3}^{(5)}),\\
\bar{\tau}_{22}^{(5)} &= \frac{5\pi}{6}(\lambda+2\mu)u_2^{(5)} + \frac{20}{\pi}\lambda\left(\frac{1}{25}e^{(0)} + \frac{1}{21}e^{(2)} + \frac{1}{9}e^{(4)} + \dots\right),\\
\bar{\tau}_{2a}^{(5)} &= \frac{5\pi}{2b}\mu u_a^{(5)} + \frac{20}{\pi}\mu\left(\frac{1}{25}u_{2,a}^{(0)} + \frac{1}{21}u_{2,a}^{(2)} + \frac{1}{9}u_{2,a}^{(4)} + \dots\right).
\end{aligned} \tag{39}
$$

6. PLATE THEORIES OF SUCCESSIVELY HIGHER ORDERS

By the truncation procedures described in Section 4 and with correction factors introduced in $\bar{U}$ and $\bar{K}$ according to Section 5, plate theories of successive orders from the zeroth up to the fourth order are obtained and will be presented in this section.

In the equations of motion for isotropic plates, the extensional motions or deformations symmetric with respect to the middle plane of the plate, such as displacement components $u_j^{(n)}$ when $n+j$ = even, are separable from the flexural or anti-symmetric motions, i.e. $u_j^{(n)}$ when $n+j$ = odd. Therefore in an Nth order theory, there are two sets of equations of motion, one for extensional motion and the other for flexural motion. In this paper the dispersion relations for extensional vibrations in even order theories (N = even) and for flexural vibrations in odd order theories (N = odd) are presented. For each order approximation, the dispersion curves for real and imaginary as well as complex wave numbers are explored in detail and compared with those from the three-dimensional theory. Since their behavior depends upon Poisson's ratio, three values, $\nu = 0{\cdot}25, 0{\cdot}30$ and $0{\cdot}35$ are used for the computation. The complex branches from the three-dimensional equation used for comparison in the present paper were obtained by Potter and Leedham [22].

Zero order theory

For the zero order approximation, according to (24) one has the energy densities

$$2\bar{U}^{(0)} = \tau_{ij}^{(0)}\varepsilon_{ij}^{(0)}, \qquad 2\bar{K}^{(0)} = \rho\alpha_2^{-p}u_i^{(0)}u_i^{(0)}, \tag{40}$$

strain components

$$\varepsilon_{ij}^{(0)} = \tfrac{1}{2}(u_{j,i}^{(0)} + u_{i,j}^{(0)}), \qquad \bar{\varepsilon}_{ij}^{(1)} = \frac{\pi}{4b}(\delta_{2i}u_j^{(1)} + \delta_{2j}u_i^{(1)}), \tag{41}$$

and stress equations of motion

$$\tau_{ij,i}^{(0)} + F_j^{(0)}/b = \rho\alpha_2^{-p}u_{j,tt}^{(0)}, \tag{42}$$

where $p = \cos^2(j\pi/2)$.

From the first equation of (30), the stress–strain relations are

$$\tau_{ij}^{(0)} = \lambda\delta_{ij}\varepsilon_{kk}^{(0)} + 2\mu\varepsilon_{ij}^{(0)} + \frac{4}{\pi}\alpha_1(\lambda\delta_{ij}\bar{\varepsilon}_{kk}^{(1)} + 2\mu\bar{\varepsilon}_{ij}^{(1)}). \tag{43}$$

By setting $\tau_{22}^{(0)} = 0$ according to (24), one can solve from (43) for $u_2^{(1)}$

$$\bar{\varepsilon}_{22}^{(2)} = \frac{\pi}{b}u_2^{(1)} = -\frac{\lambda}{\lambda+2\mu}\left(\frac{\pi}{4\alpha_1}\right)(\varepsilon_{11}^{(0)} + \varepsilon_{33}^{(1)}). \tag{44}$$

Substitution of (44) back into (43) yields the stress–strain relations for the zero order approximation

$$\tau_{ij}^{(0)} = \lambda'\delta_{ij}(\varepsilon_{11}^{(0)} + \varepsilon_{33}^{(0)}) + 2\mu\left[\varepsilon_{ij}^{(0)} - \frac{\lambda}{\lambda+2\mu}\delta_{2i}\delta_{2j}(\varepsilon_{11}^{(0)} + \varepsilon_{33}^{(0)})\right] \tag{45}$$

or

$$\begin{aligned}
\tau_{11}^{(0)} &= \lambda'(\varepsilon_{11}^{(0)} + \varepsilon_{33}^{(0)}) + 2\mu\varepsilon_{11}^{(0)},\\
\tau_{33}^{(0)} &= \lambda'(\varepsilon_{11}^{(0)} + \varepsilon_{33}^{(0)}) + 2\mu\varepsilon_{33}^{(0)},\\
\tau_{12}^{(0)} &= 2\mu\varepsilon_{12}^{(0)}, \qquad \tau_{23}^{(0)} = 2\mu\varepsilon_{23}^{(0)}, \qquad \tau_{31}^{(0)} = 2\mu\varepsilon_{31}^{(0)},
\end{aligned} \tag{46}$$

where $\lambda' = 2\mu\lambda/(\lambda+2\mu)$. The displacement equations of motion for zero order approximations can be obtained by inserting (46) into (42):

$$\begin{aligned}
\mu\nabla^2 u_a^{(0)} + (\lambda'+\mu)\,\mathrm{e}_{,a}^{(0)} + F_a^{(0)}/b &= \rho u_{a,tt}^{(0)}\\
\mu\nabla^2 u_2^{(0)} + F_2^{(0)}/b &= \rho u_{2,tt}^{(0)}/\alpha_2
\end{aligned} \tag{47}$$

where $a = 1, 3$ and $\nabla^2 = (\partial_1^2 + \partial_3^2)$.

Equations (47) are equivalent to Mindlin's zero order plate equations [1] while the first two in (47) are equivalent to the classical extensional theory [5] for thin plates (by Poisson [2] and by Cauchy [3]). The third equation, as pointed out by Mindlin, is not useful by itself and should be included into the first order flexural theory.

Consider a straight-crested wave propagating in the x_1 direction:

$$u_1^{(0)} = A\,\mathrm{e}^{i(\xi x_1 - \omega t)}, \qquad u_2^{(0)} = u_3^{(0)} = 0.$$

Inserting the above into (47) with $F_j^{(0)} = 0$, one finds

$$\omega = \xi\sqrt{[E/\rho(1-\nu^2)]} \quad \text{or} \quad \Omega = z\sqrt{[2/(1-\nu)]} \tag{48}$$

which agrees with the slope of the lowest extensional branch in the three-dimensional theory as both ω and ξ approach zero.

If one considers *SH* waves or face-shear vibrations by setting $F_j^{(0)} = 0$ and

$$u_1^{(0)} = u_2^{(0)} = 0, \qquad u_3^{(0)} = B\,\mathrm{e}^{i(\xi x_1 - \omega t)}$$

in (47), one obtains

$$\omega = \xi\sqrt{(\mu/\rho)} \quad \text{or} \quad \Omega = z \tag{49}$$

which coincides with the result for *SH* waves in the three-dimensional theory.

First order theory

According to the truncation procedure given by (25), for first order theory one sets $N = 1$; then only the displacement, stress and strain components of orders less than or equal to 1 ($=N$) will be retained. Thus, the energy densities become

$$
\begin{aligned}
2\bar{U}^{(1)} &= \tau_{ij}^{(0)}\varepsilon_{ij}^{(0)} + \tau_{ij}^{(1)}\varepsilon_{ij}^{(1)} + \bar{\tau}_{ij}^{(1)}\bar{\varepsilon}_{ij}^{(1)} \\
2\bar{K}^{(1)} &= \rho(\alpha_2^{-p} u_{i,t}^{(0)} u_{i,t}^{(0)} + u_{i,t}^{(1)} u_{i,t}^{(1)})
\end{aligned}
\tag{50}
$$

and the stress equations of motion are

$$
\begin{aligned}
\tau_{ij,i}^{(0)} + \frac{1}{b}F_j^{(0)} &= \rho\alpha_2^{-p} u_{j,tt}^{(0)} \\
\tau_{ij,i}^{(1)} + \frac{\pi}{2b}\bar{\tau}_{2j}^{(1)} + \frac{1}{b}F_j^{(1)} &= \rho u_{j,tt}^{(1)}.
\end{aligned}
\tag{51}
$$

The stress–strain relations and the stress–displacement relations for the first order approximation are readily obtained from (30), (31) and (34), (35), respectively, by discarding those components $\varepsilon_{ij}^{(n)}$, $\bar{\varepsilon}_{ij}^{(n)}$ and $u_j^{(n)}$ for which $n > 1$. The displacement equations of motion of the first order theory can be obtained by inserting the stress–displacement relations into (51). Thus for the flexural motion, $a = 1,3$:

$$
\begin{aligned}
\mu\nabla^2 u_2^{(0)} + \frac{2\mu}{b}\alpha_1\,\mathrm{e}^{(1)} + \frac{1}{b}F_2^{(0)} &= \frac{\rho}{\alpha_2} u_{2,tt}^{(0)} \\
\mu\nabla^2 u_a^{(1)} + (\lambda+\mu)\,e_{,a}^{(1)} - \mu\left(\frac{\pi}{2b}\right)^2 u_a^{(1)} - \frac{2\mu}{b}\alpha_1 u_{2,a}^{(0)} + \frac{1}{b}F_a^{(1)} &= \rho u_{a,tt}^{(1)}
\end{aligned}
\tag{52}
$$

and for extensional motion, $a = 1,3$:

$$
\begin{aligned}
\mu\nabla^2 u_a^{(0)} + (\lambda+\mu)\,\mathrm{e}_{,a}^{(0)} + \frac{2\lambda}{b}\alpha_1 u_{2,a}^{(1)} + \frac{1}{b}F_a^{(0)} &= \rho u_{a,tt}^{(0)} \\
\mu\alpha_2\nabla^2 u_2^{(1)} - (\lambda+2\mu)\left(\frac{\pi}{2b}\right)^2 u_2^{(1)} - \frac{2\lambda}{b}\alpha_1\,\mathrm{e}^{(0)} + \frac{1}{b}F_2^{(1)} &= \rho u_{2,tt}^{(1)}.
\end{aligned}
\tag{53}
$$

Equations (52) and (53) are equivalent to Mindlin's first order plate theory [1]. Similar equations for flexural theory (52) were obtained by Mindlin [6], Uflyand [7], Reissner [8, 9], Timoshenko [11, 12] and Bresse [10], while equations similar to (53) for extensional theory were obtained by Kane and Mindlin [13].

In the flexural equations of the first order approximation (52), one may consider straight-crested free waves by setting

$$
u_2^{(0)} = A_2^{(0)}\,\mathrm{e}^{i(\xi x_1 - \omega t)}, \qquad u_1^{(1)} = -iA_1^{(1)}\,\mathrm{e}^{i(\xi x_1 - \omega t)}, \qquad u_3^{(1)} = 0
$$
$$
F_2^{(0)} = F_1^{(1)} = F_3^{(1)} = 0
$$

and obtains the dispersion relation, in dimensionless form, as

$$
\begin{vmatrix}
z^2 - \alpha_2^{-1}\Omega^2 & \alpha_1\dfrac{4}{\pi}z \\
\alpha_1\dfrac{4}{\pi}z & k^2z^2 + 1 - \Omega^2
\end{vmatrix} = 0.
\tag{54}
$$

Expanding the above and setting $z = 0$, one finds

$$\Omega = 0, 1.$$

The second root $\Omega = 1$ or $\omega = \omega_1 = \pi v_2/2b$ is the resonant frequency of the lowest simple thickness-shear mode and also the "exact" cutoff frequency at zero wave number for the second flexural branch.

For $\Omega \ll 1$ and $|z| < 1$, (54) reduces to

$$k^2 z^4 - \alpha_2^{-1}\Omega^2 + \left[1 - \left(\frac{4}{\pi}\right)^2 \alpha_1^2\right] z^2 = 0.$$

In order to have $z = 0$ as Ω approaches zero, one must set

$$\alpha_1 = \pi/4.$$

Then $\Omega = k\alpha_2^{1/2}z^2$ which, however, does not agree with the limit $\Omega = z^2\sqrt{[(1-k^{-2})/3]}$ from three-dimensional theory. Therefore, the lowest flexural branch has the correct slope but incorrect curvature at $\Omega = 0$ and $z = 0$ if α_1 is set equal to $\pi/4$.

For $\Omega > 1$ and $z > 1$, the phase velocity of the lowest flexural branch approaches, from (54), the value

$$\frac{\Omega}{z} = \frac{v}{v_2} = \alpha_2.$$

Hence the phase velocity of the lowest flexural branch will approach that of the Rayleigh surface waves, the exact value, if α_2 is set equal to the values given in Section 5.

The dispersion curves for the first order flexural theory are computed and compared with those from the Rayleigh–Lamb frequency equation as shown in Figs. 3(a)–(c). It can be seen that the two sets of curves match quite well for $\Omega \leq 1{\cdot}5$; the incorrect curvature at $\Omega = 0$, $z = 0$ appears to have no observable effect upon the numerical values.

By setting, in (52), $F_j^{(n)} = 0$ and

$$u_1^{(1)} = u_2^{(0)} = 0, \qquad u_3^{(1)} = A_3^{(1)}\,e^{i(\xi x_1 - \omega t)}$$

one obtains

$$\Omega^2 = z^2 + 1$$

which agrees with the exact result for face-shear vibrations in the three-dimensional theory.

Second order theory

Following the truncation procedure described in Section 5 by setting $N = 2$ in (25), the energy densities for the second order theory are

$$2\overline{U}^{(2)} = \tau_{ij}^{(0)}\varepsilon_{ij}^{(0)} + \tau_{ij}^{(1)}\varepsilon_{ij}^{(1)} + \tau_{ij}^{(2)}\varepsilon_{ij}^{(2)} + \bar{\tau}_{ij}^{(1)}\bar{\varepsilon}_{ij}^{(1)} + \bar{\tau}_{ij}^{(2)}\bar{\varepsilon}_{ij}^{(2)},$$

$$2\overline{K}^{(2)} = \rho(\alpha_2^{-p}u_{i,t}^{(0)}u_{i,t}^{(0)} + u_{i,t}^{(1)}u_{i,t}^{(1)} + u_{i,t}^{(2)}u_{i,t}^{(2)}), \tag{55}$$

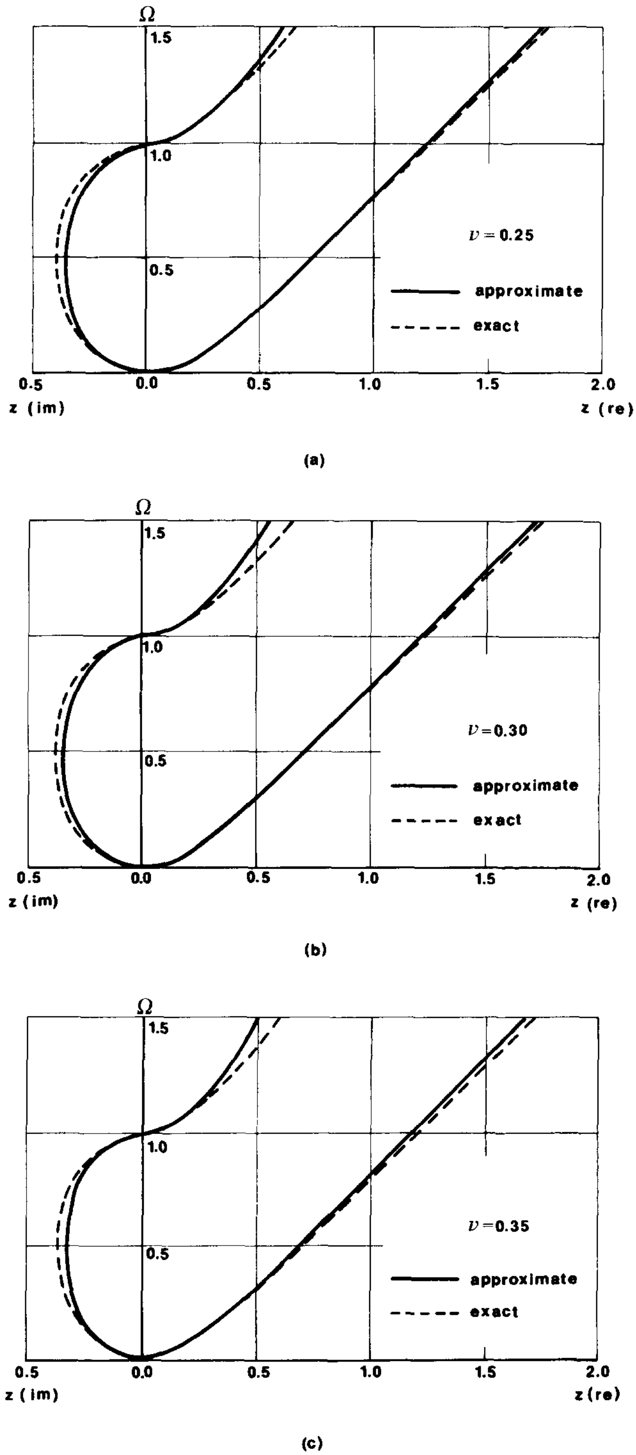


FIG. 3. Dispersion curves for the first order flexural theory.

and the stress equations of motion are

$$
\begin{aligned}
\tau_{ij,i}^{(0)}+\frac{1}{b}F_j^{(0)} &= \rho\alpha_2^{-p}u_{j,tt}^{(0)}, \\
\tau_{ij,i}^{(1)}+\frac{\pi}{2b}\bar{\tau}_{2j}^{(1)}+\frac{1}{b}F_j^{(1)} &= \rho u_{j,tt}^{(1)}, \\
\tau_{ij,i}^{(2)}+\frac{\pi}{b}\bar{\tau}_{2j}^{(2)}+\frac{1}{b}F_j^{(2)} &= \rho u_{j,tt}^{(2)}.
\end{aligned}
\tag{56}
$$

The stress–strain relations and stress–displacement relations are given by (30), (31) and (34)–(36), respectively, with strain and displacement components of orders higher than two discarded. By direct substitution, the displacement equations of motion of the second order theory are obtained; for flexural motion ($a = 1,3$):

$$
\begin{aligned}
&\mu\nabla^2 u_2^{(0)}+\frac{2\mu}{b}\alpha_1\,\mathrm{e}^{(1)}+\frac{1}{b}F_2^{(0)} = \frac{\rho}{\alpha_2}u_{2,tt}^{(0)} \\
&\mu\nabla^2 u_a^{(1)}+(\lambda+\mu)\,e_{,a}^{(1)}-\mu\left(\frac{\pi}{2b}\right)^2 u_a^{(1)}-\frac{2\mu}{b}\alpha_1 u_{2,a}^{(0)}+\frac{2(4\lambda+\mu)}{3b}u_{2,a}^{(2)}+\frac{1}{b}F_a^{(1)} = \rho u_{a,tt}^{(1)} \\
&\mu\nabla^2 u_2^{(1)}-(\lambda+2\mu)\left(\frac{\pi}{b}\right)^2 u_2^{(2)}-\frac{2(4\lambda+\mu)}{3b}\,\mathrm{e}^{(1)}+\frac{1}{b}F_2^{(2)} = \rho u_{2,tt}^{(2)}
\end{aligned}
\tag{57}
$$

and for extensional motion ($a = 1,3$):

$$
\begin{aligned}
&\mu\nabla^2 u_a^{(0)}+(\lambda+\mu)\,e_{,a}^{(0)}+\frac{2\lambda}{b}\alpha_1 u_{2,a}^{(1)}+\frac{1}{b}F_a^{(0)} = \rho u_{a,tt}^{(0)} \\
&\mu\alpha_2\nabla^2 u_2^{(1)}-(\lambda+2\mu)\left(\frac{\pi}{2b}\right)^2 u_2^{(1)}-\frac{2\lambda}{b}\alpha_1\,\mathrm{e}^{(0)}+\frac{2(\lambda+4\mu)}{3b}\,\mathrm{e}^{(2)}+\frac{1}{b}F_2^{(1)} = \rho u_{2,tt}^{(1)} \\
&\mu\nabla^2 u_a^{(2)}+(\lambda+\mu)\,e_{,a}^{(2)}-\mu\left(\frac{\pi}{2b}\right)^2 u_1^{(2)}-\frac{2(\lambda+4\mu)}{3b}u_{2,a}^{(1)}+\frac{1}{b}F_a^{(2)} = \rho u_{a,tt}^{(2)}.
\end{aligned}
\tag{58}
$$

Equations (58) are equivalent to Mindlin and Medick's [14] extensional plate equations of second order approximation and are closely related to Reissner's equations of equilibrium [23] for plates.

Consider straight-crested waves propagating in the x_1 direction by setting

$$
\begin{aligned}
u_1^{(0)} &= A_1^{(0)}\,\mathrm{e}^{i(\xi x_1-\omega t)}, \qquad u_3^{(0)} = 0 \\
u_2^{(1)} &= -iA_2^{(1)}\,\mathrm{e}^{i(\xi x_1-\omega t)} \\
u_1^{(2)} &= A_1^{(2)}\,\mathrm{e}^{i(\xi x_1-\omega t)}, \qquad u_3^{(2)} = 0 \\
F_1^{(0)} &= F_3^{(0)} = F_2^{(1)} = F_1^{(2)} = F_3^{(2)} = 0
\end{aligned}
$$

in (58), then the dispersion relation is obtained as

$$\begin{vmatrix} k^2z^2-\Omega^2 & \alpha_1\dfrac{4}{\pi}(k^2-2)z & 0 \\ \alpha_1\dfrac{4}{\pi}(k^2-2)z & k^2+\alpha_2z^2-\Omega^2 & -\dfrac{4}{3\pi}(k^2+2)z \\ 0 & -\dfrac{4}{3\pi}(k^2+2)z & k^2z^2+4-\Omega^2 \end{vmatrix} = 0. \tag{59}$$

At $z = 0$, (59) gives the "exact" cut-off frequencies

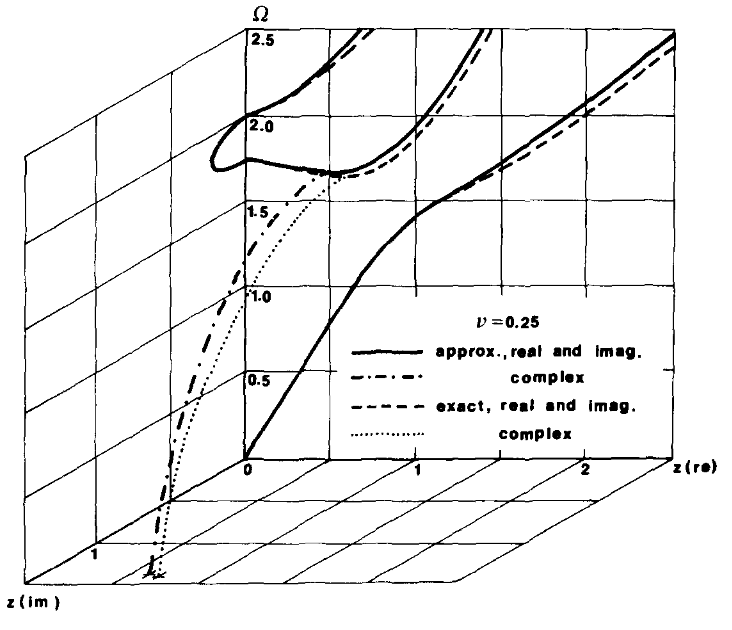
$$\Omega = 0, k, 2.$$

When both z and $\Omega \ll 1$, (59) reduces to (48), the "exact" limit, if $\alpha_1 = \pi/4$. For $z > 1$ and $\Omega > 1$, one finds from (59) that the phase velocity of the lowest extensional branch approaches the value

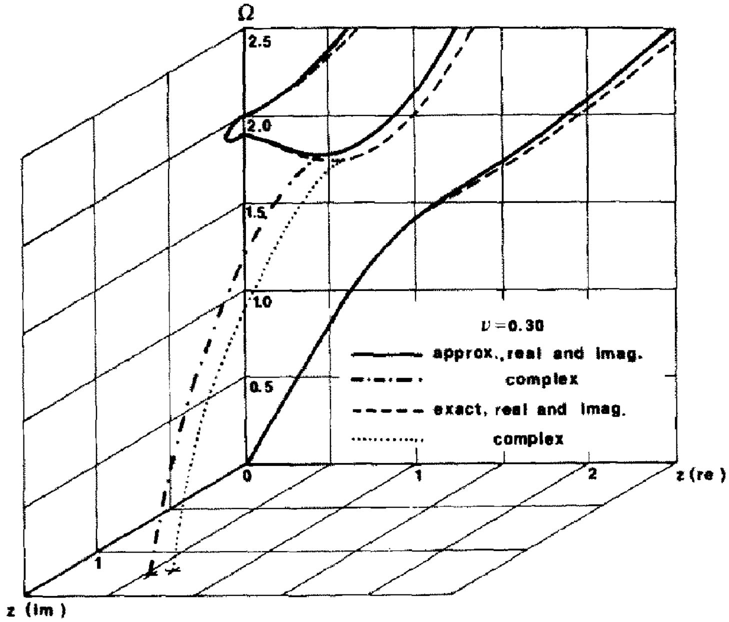
$$\frac{\Omega}{z} = \frac{v}{v_2} = \alpha_2$$

which, according to Section 5, is the phase velocity for Rayleigh surface waves and also the "exact" limit of the phase velocity for the lowest extensional branch as both Ω and z become large.

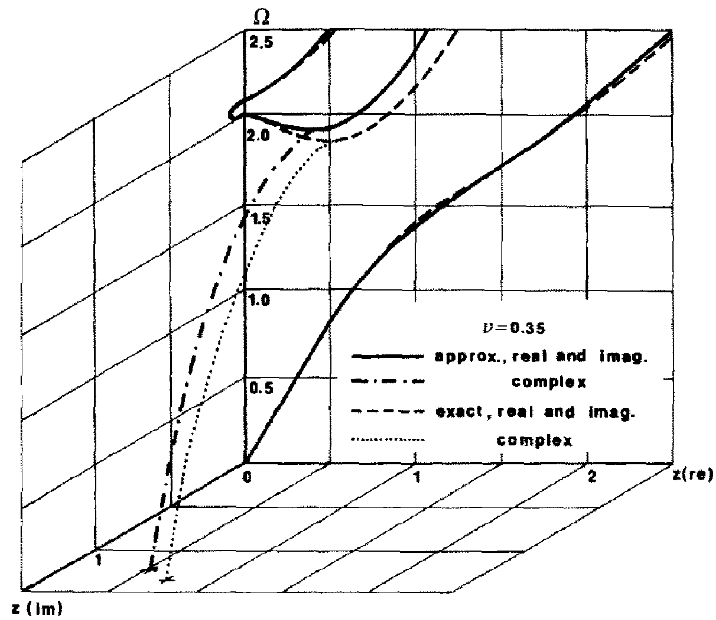
The dispersion curves are presented and compared with the results from three-dimensional theory in Figs. 4(a)–(c).



(a)



(b)



(c)

FIG. 4. Dispersion curves for the second order extensional theory.

At this point one may see that the same value, $\alpha_1 = \pi/4$, is used for correcting the slopes of both the lowest flexural and extensional branches at $\Omega = 0$, $z = 0$ and the same value for α_2 is used for correcting the phase velocities of the same two branches when the values of Ω and z become large. No additional correction is needed in the higher branches in the subsequent higher order theories as will be seen in the following subsections.

For face-shear vibrations (58) yields, in a similar manner, the exact results

$$\Omega^2 = z^2 + n^2, \qquad n = 0, 2.$$

Third order theory

Energy densities:

$$\begin{aligned} 2\bar{U}^{(3)} &= \tau_{ij}^{(0)}\varepsilon_{ij}^{(0)} + \tau_{ij}^{(1)}\varepsilon_{ij}^{(1)} + \dots \tau_{ij}^{(3)}\varepsilon_{ij}^{(3)} + \bar{\tau}_{ij}^{(1)}\bar{\varepsilon}_{ij}^{(1)} + \dots \bar{\tau}_{ij}^{(3)}\bar{\varepsilon}_{ij}^{(3)} \\ 2\bar{K}^{(3)} &= \rho(\alpha_2^{-p} u_{i,t}^{(0)}u_{i,t}^{(0)} + u_{i,t}^{(1)}u_{i,t}^{(1)} + \dots u_{i,t}^{(3)}u_{i,t}^{(3)}). \end{aligned} \tag{60}$$

Stress equations of motion:

$$\begin{aligned} \tau_{ij,i}^{(0)} + \frac{1}{b}F_j^{(0)} &= \rho\alpha_2^{-p} u_{j,tt}^{(0)} \\ \tau_{ij,i}^{(n)} + \frac{n\pi}{2b}\bar{\tau}_{2j}^{(n)} + \frac{1}{b}F_j^{(n)} &= \rho u_{j,tt}^{(n)}, \qquad n = 1, 2, 3. \end{aligned} \tag{61}$$

The stress–strain relations and stress–displacement relations are given by (30)–(31) and (34)–(37), respectively, with components of strain and displacement of order higher than three discarded.

Displacement equations of motion (for flexural motion, $a = 1, 3$):

$$\begin{aligned} &\mu\nabla^2 u_2^{(0)} + \frac{2\mu}{b}\alpha_1\, e^{(1)} + \frac{2\mu}{b}\, e^{(3)} + \frac{1}{b}F_2^{(0)} = \frac{\rho}{\alpha_2} u_{2,tt}^{(0)} \\ &\mu\nabla^2 u_a^{(1)} + (\lambda+\mu)\, e_{,a}^{(1)} - \mu\left(\frac{\pi}{2b}\right)^2 u_a^{(1)} - \frac{2\mu}{b}\alpha_1 u_{2,a}^{(0)} + \frac{2(4\lambda+\mu)}{3b}u_{2,a}^{(2)} + \frac{F_2^{(1)}}{b} = \rho u_{a,tt}^{(1)} \\ &\mu\nabla^2 u_2^{(2)} - (\lambda+2\mu)\left(\frac{\pi}{2b}\right)^2 u_2^{(2)} - \frac{2(4\lambda+\mu)}{3b}\, e^{(1)} + \frac{2(4\lambda+9\mu)}{5b}\, e^{(3)} + \frac{F_2^{(2)}}{b} = \rho u_{2,tt}^{(2)} \\ &\mu\nabla^2 u_a^{(3)} + (\lambda+\mu)\, e_{,a}^{(3)} - \mu\left(\frac{3\pi}{2b}\right)^2 u_a^{(3)} - \frac{2\mu}{b}u_{2,a}^{(0)} - \frac{2(4\lambda+9\mu)}{5b}u_{2,a}^{(3)} + \frac{F_a^{(3)}}{b} = \rho u_{a,tt}^{(3)}. \end{aligned} \tag{62}$$

For extensional motion ($a = 1, 3$):

$$\begin{aligned} &\mu\nabla^2 u_a^{(0)} + (\lambda+\mu)\, e_{,a}^{(0)} + \frac{2\lambda}{b}\alpha_1 u_{2,a}^{(1)} + \frac{2\lambda}{b}u_{2,a}^{(3)} + \frac{F_a^{(0)}}{b} = \rho u_{a,tt}^{(0)} \\ &\mu\alpha_2\nabla^2 u_2^{(1)} - (\lambda+2\mu)\left(\frac{\pi}{2b}\right)^2 u_2^{(1)} - \frac{2\lambda}{b}\alpha_1\, e^{(0)} + \frac{2(\lambda+4\mu)}{3b}\, e^{(2)} + \frac{2(\lambda+16\mu)}{15b}\, e^{(4)} + \frac{F_2^{(1)}}{b} = \rho u_{2,tt}^{(1)} \\ &\mu\nabla^2 u_a^{(2)} + (\lambda+\mu)\, e_{,a}^{(2)} - \mu\left(\frac{\pi}{2b}\right)^2 u_a^{(2)} - \frac{2(\lambda+4\mu)}{3b}u_{2,a}^{(1)} + \frac{2(9\lambda+4\mu)}{5b}u_{2,a}^{(3)} + \frac{F_a^{(2)}}{b} = \rho u_{a,tt}^{(2)} \\ &\mu\nabla^2 u_2^{(3)} - (\lambda+2\mu)\left(\frac{3\pi}{2b}\right)^2 u_2^{(3)} - \frac{2\lambda}{b}\, e^{(0)} - \frac{2(9\lambda+4\mu)}{5b}\, e^{(2)} + \frac{F_2^{(3)}}{b} = \rho u_{2,tt}^{(3)}. \end{aligned} \tag{63}$$

By setting $F_j^{(n)} = 0$, $u_3^{(1)} = u_3^{(3)} = 0$ and employing

$$u_2^{(0)} = A_2^{(0)}\,e^{i(\xi x_1 - \omega t)}, \qquad u_2^{(2)} = A_2^{(2)}\,e^{i(\xi x_1 - \omega t)}$$
$$u_1^{(1)} = -iA_1^{(1)}\,e^{i(\xi x_1 - \omega t)}, \quad u_1^{(3)} = -iA_1^{(3)}\,e^{i(\xi x_1 - \omega t)}$$

in (62), one obtains the dispersion relation as

$$\begin{vmatrix} z^2-\alpha_2^{-1}\Omega^2 & \alpha_1\frac{4}{\pi}z & 0 & \frac{4}{\pi}z \\ \alpha_1\frac{4}{\pi}z & k^2z^2+1-\Omega^2 & -\frac{4}{3\pi}(4k^2-7)z & 0 \\ 0 & -\frac{4}{3\pi}(4k^2-7)z & z^2+4k^2-\Omega^2 & \frac{4}{5\pi}(4k^2+1)z \\ \frac{4}{\pi}z & 0 & \frac{4}{5\pi}(4k^2+1)z & k^2z^2+9-\Omega^2 \end{vmatrix} = 0. \tag{64}$$

Dispersion curves computed from (64) and those from three-dimensional theory are presented in Figs. 5(a)–(c). It can be seen that the lower branches match with the "exact" curves better than the corresponding ones in the first order flexural theory, since the higher order theory not only accommodates additional branches at higher frequencies but also improves the accuracy for the lower branches. The behavior of the complex branch varies with the values of Poisson's ratio. The discrepancy for the complex branch is mostly due to the difference in the real parts. The matching for the imaginary parts is very close. The mode shape corresponding to a complex wave number can be expressed as the product of a trigonometrical function (depending on the real part of the wave number) and a hyperbolic function (depending on the imaginary part of the wave number). Hence the mode has the shape of an oscillatory variation within the envelope of the hyperbolic function. Therefore in the case of matching the complex branches it is more important to have a close agreement of the imaginary parts than of the real parts, since the former control the amplitude of the mode.

As for the *SH* waves, in a similar manner (62) yields the exact results

$$\Omega^2 = z^2+n^2,\ n = 1, 3.$$

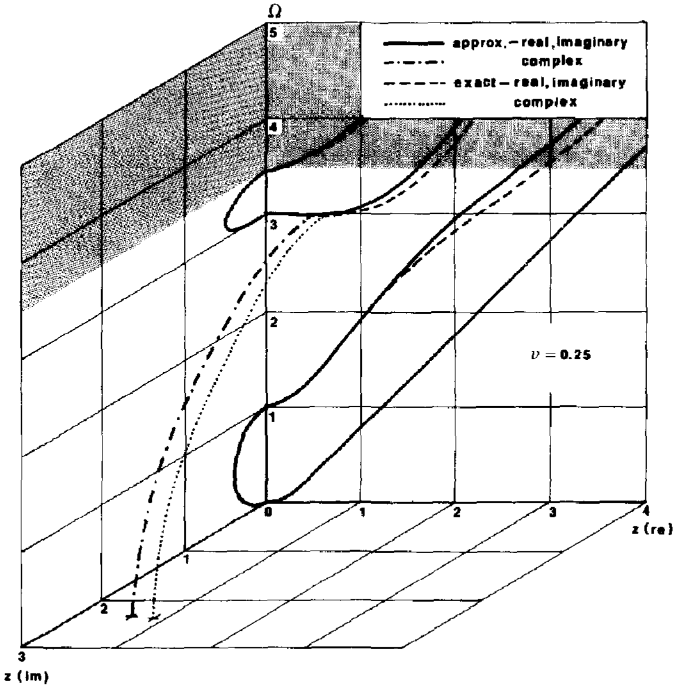
Fourth order theory

Energy densities:

$$\begin{aligned} 2\bar{U}^{(4)} &= \tau_{ij}^{(0)}\varepsilon_{ij}^{(0)} + \tau_{ij}^{(1)}\varepsilon_{ij}^{(1)} + \dots + \tau_{ij}^{(4)}\varepsilon_{ij}^{(4)} + \bar{\tau}_{ij}^{(1)}\bar{\varepsilon}_{ij}^{(1)} \\ &\quad + \bar{\tau}_{ij}^{(2)}\bar{\varepsilon}_{ij}^{(2)} + \dots + \bar{\tau}_{ij}^{(4)}\bar{\varepsilon}_{ij}^{(4)} \\ 2\bar{K}^{(4)} &= \rho(\alpha_2^{-p}u_{i,t}^{(0)}u_{i,t}^{(0)} + u_{i,t}^{(1)}u_{i,t}^{(1)} + \dots + u_{i,t}^{(4)}u_{i,t}^{(4)}). \end{aligned} \tag{65}$$

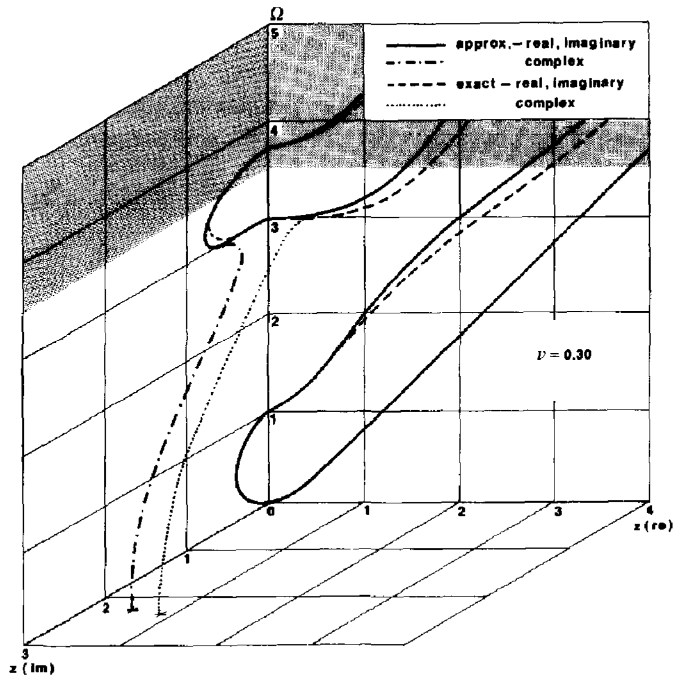
Stress equations of motion:

$$\begin{aligned} \tau_{ij,i}^{(0)} + \frac{1}{b}F_j^{(0)} &= \rho\alpha_2^{-p}u_{j,tt}^{(0)} \\ \tau_{ij,i}^{(n)} + \frac{n\pi}{2b}\bar{\tau}_{2j}^{(n)} + \frac{1}{b}F_j^{(n)} &= \rho u_{j,tt}^{(n)}, \qquad n = 1, 2, 3, 4. \end{aligned} \tag{66}$$



Ω
approx. – real, imaginary
complex
exact – real, imaginary
complex
$\nu = 0.25$
z (re)
z (im)

(a)



Ω
approx. – real, imaginary
complex
exact – real, imaginary
complex
$\nu = 0.30$
z (re)
z (im)

(b)

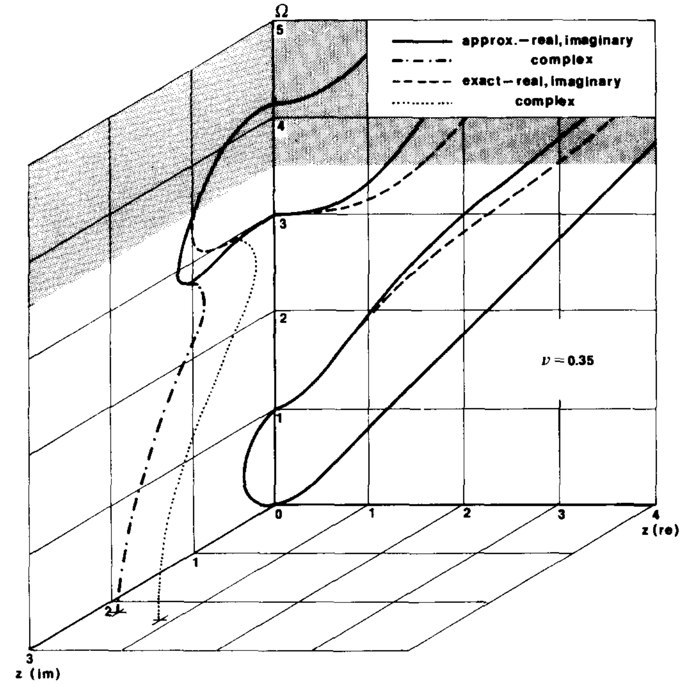


FIG. 5. Dispersion curves for the third order flexural theory.

The stress–strain relations and stress–displacement relations can be obtained from (30), (31) and (34)–(38), respectively, with components of strain and displacement of order higher than four discarded.

Displacement equations of motion (for flexural motion, $a = 1, 3$):

$$
\begin{aligned}
&\mu\nabla^2 u_2^{(0)} + \frac{2\mu}{b}\alpha_1\, e^{(1)} + \frac{2\mu}{b} e^{(3)} + \frac{1}{b}F_2^{(0)} = \frac{\rho}{\alpha_2}u_{2,tt}^{(0)} \\
&\mu\nabla^2 u_a^{(1)} + (\lambda+\mu)\, e_{,a}^{(1)} - \mu\left(\frac{\pi}{2b}\right)^2 u_a^{(1)} - \frac{2\mu}{b}\alpha_1 u_{2,a}^{(0)} + \frac{2(4\lambda+\mu)}{3b}u_{2,a}^{(2)} \\
&\quad + \frac{2(16\lambda+\mu)}{15b}u_{2,a}^{(4)} + \frac{1}{b}F_a^{(1)} = \rho u_{a,tt}^{(1)} \\
&\mu\nabla^2 u_2^{(2)} - (\lambda+2\mu)\left(\frac{\pi}{b}\right)^2 u_2^{(2)} - \frac{2(4\lambda+\mu)}{3b} e^{(1)} + \frac{2(4\lambda+9\mu)}{5b} e^{(3)} \\
&\quad + \frac{1}{b}F_2^{(2)} = \rho u_{2,tt}^{(2)} \\
&\mu\nabla^2 u_a^{(3)} + (\lambda+\mu)\, e_{,a}^{(3)} - \mu\left(\frac{3\pi}{2b}\right)^2 u_a^{(3)} - \frac{2\mu}{b}u_{2,a}^{(0)} - \frac{2(4\lambda+9\mu)}{5b}u_{2,a}^{(2)} \\
&\quad + \frac{2(16\lambda+9\mu)}{7b}u_{2,a}^{(4)} + \frac{1}{b}F_a^{(3)} = \rho u_{a,tt}^{(3)}
\end{aligned}
\tag{67}
$$

$$\mu\nabla^2 u_2^{(4)} - (\lambda+2\mu)\left(\frac{2\pi}{b}\right)^2 u_2^{(4)} - \frac{2(16\lambda+\mu)}{15b}\,\mathrm{e}^{(1)} - \frac{2(16\lambda+9\mu)}{7b}\,\mathrm{e}^{(3)} + \frac{1}{b}F_2^{(4)}$$
$$= \rho u_{2,tt}^{(4)}.$$

For extensional motion ($a = 1, 3$):

$$\mu\nabla^2 u_a^{(0)} + (\lambda+\mu)\,e_{,a}^{(0)} + \frac{2\lambda}{b}\alpha_1 u_{2,a}^{(1)} + \frac{2\lambda}{b}u_{2,a}^{(3)} + \frac{1}{b}F_a^{(0)} = \rho u_{a,tt}^{(0)}$$

$$\mu\alpha_2\nabla^2 u_2^{(1)} - (\lambda+2\mu)\left(\frac{\pi}{2b}\right)^2 u_2^{(1)} - \frac{2\lambda}{b}\alpha_1\,\mathrm{e}^{(0)} + \frac{2(\lambda+4\mu)}{3b}\,\mathrm{e}^{(2)} + \frac{2(\lambda+16\mu)}{15b}\,\mathrm{e}^{(4)}$$
$$+\frac{1}{b}F_2^{(1)} = \rho u_{2,tt}^{(1)}$$

$$\mu\nabla^2 u_a^{(2)} + (\lambda+\mu)\,e_{,a}^{(2)} - \mu\left(\frac{\pi}{b}\right)^2 u_a^{(2)} - \frac{2(\lambda+4\mu)}{3b}u_{2,a}^{(1)} + \frac{2(9\lambda+4\mu)}{5b}u_{2,a}^{(3)}$$
$$+\frac{1}{b}F_a^{(2)} = \rho u_{a,tt}^{(2)} \tag{68}$$

$$\mu\nabla^2 u_2^{(3)} - (\lambda+2\mu)\left(\frac{3\pi}{2b}\right)^2 u_2^{(3)} - \frac{2\lambda}{b}\,\mathrm{e}^{(0)} - \frac{2(9\lambda+4\mu)}{5b}\,\mathrm{e}^{(2)} + \frac{2(9\lambda+16\mu)}{7b}\,\mathrm{e}^{(4)}$$
$$+\frac{1}{b}F_2^{(3)} = \rho u_{2,tt}^{(3)}$$

$$\mu\nabla^2 u_a^{(4)} + (\lambda+\mu)\,e_{,a}^{(4)} - \mu\left(\frac{2\pi}{b}\right)^2 u_a^{(4)} - \frac{2(\lambda+16\mu)}{15b}u_{2,a}^{(1)} - \frac{2(9\lambda+16\mu)}{7b}u_{2,a}^{(3)}$$
$$+\frac{1}{b}F_a^{(4)} = \rho u_{a,tt}^{(4)}.$$

By setting $F_j^{(n)} = 0$, $u_3^{(0)} = u_3^{(2)} = u_3^{(4)} = 0$ and employing

$$u_1^{(0)} = A_1^{(0)}\,\mathrm{e}^{i(\xi x_1 - \omega t)},\; u_1^{(2)} = A_1^{(2)}\,\mathrm{e}^{i(\xi x_1 - \omega t)},\; u_1^{(4)} = A_1^{(4)}\,\mathrm{e}^{i(\xi x_1 - \omega t)}$$
$$u_2^{(1)} = -iA_2^{(1)}\,\mathrm{e}^{i(\xi x_1 - \omega t)},\; u_2^{(3)} = -iA_2^{(3)}\,\mathrm{e}^{i(\xi x_1 - \omega t)}$$

in (68), one obtains the dispersion relation for extensional motion as

$$\begin{vmatrix} k^2z^2-\Omega^2 & \alpha_1\frac{4}{\pi}(k^2-2)z & 0 & \frac{4}{\pi}(k^2-2)z & 0 \\ \alpha_1\frac{4}{\pi}(k^2-2)z & k^2+\alpha_2 z^2-\Omega^2 & \frac{-4}{3\pi}(k^2+2)z & 0 & \frac{-4}{15\pi}(k^2+14)z \\ 0 & \frac{-4}{3\pi}(k^2+2)z & k^2z^2+4-\Omega^2 & \frac{4}{5\pi}(9k^2-14)z & 0 \\ \frac{4}{\pi}(k^2-2)z & 0 & \frac{4}{5\pi}(9k^2-14)z & 9k^2+z^2-\Omega^2 & \frac{-4}{7\pi}(9k^2-2)z \\ 0 & \frac{-4}{15\pi}(k^2+14)z & 0 & \frac{-4}{7\pi}(9k^2-2)z & k^2z^2+16-\Omega^2 \end{vmatrix} = 0. \tag{69}$$

Dispersion curves computed from (69) are compared with the exact ones as shown in Figs. 6(a)–(c).

For SH waves in an infinite plate, the dispersion relation obtained from (68) is

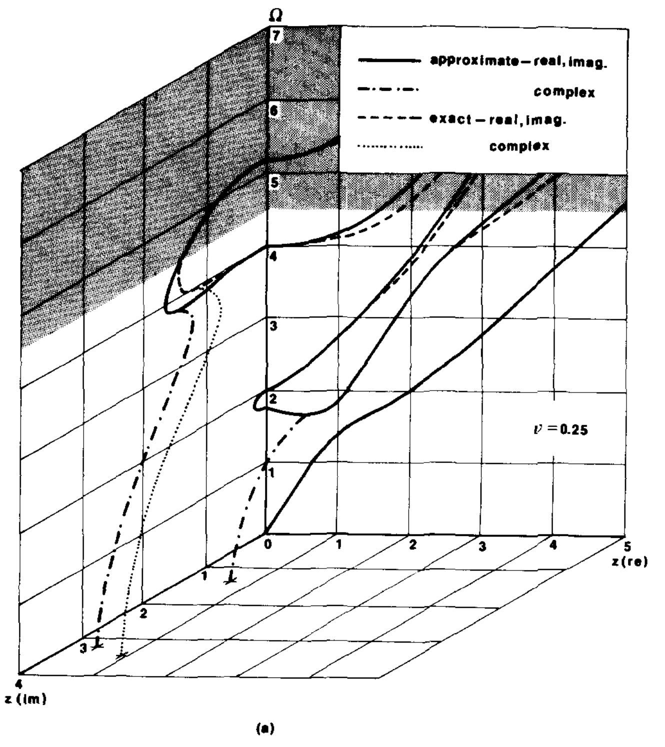
$$\Omega^2 = z^2+n^2,\ n = 0, 2, 4$$

which agrees with the exact results.

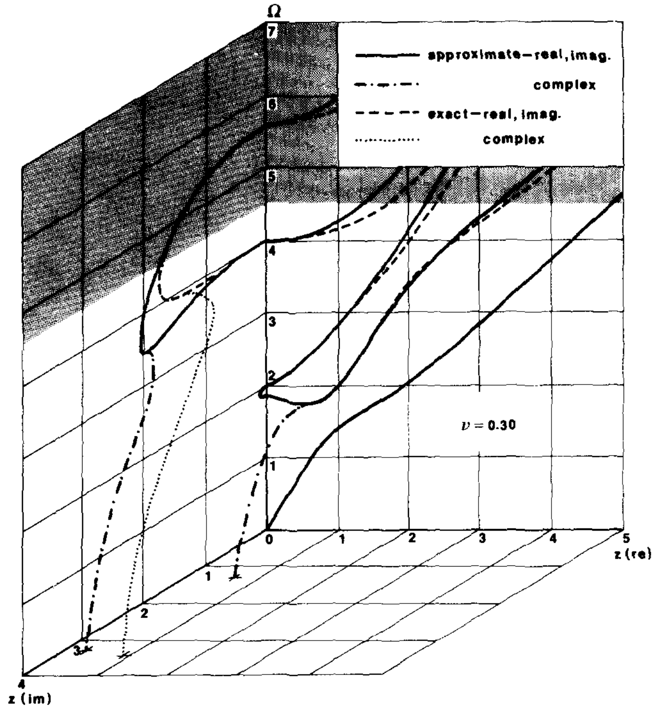
7. GENERATION OF HIGHER ORDER DISPERSION RELATIONS

It has been demonstrated that by the truncation procedure given in (25) one may generate an Nth order approximate theory (any $N > 0$) for plates in a systematic manner and no additional correction coefficients, aside from α_1 and α_2, being needed for any higher order theory. For the Nth order theory the dispersion relation for straight-crested wave propagation in the x_1 direction may be obtained by substitution of appropriate wave form solutions into the displacement equations of motion in a straightforward manner. However, when the order N is large, the process becomes tedious. A general method for generation of dispersion relation for an Nth order theory is described as follows.

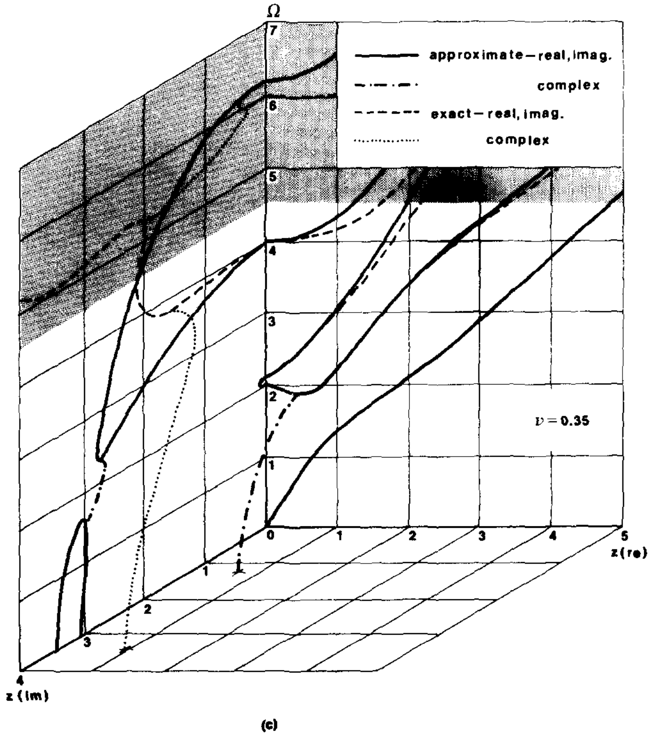
By inspecting dispersion relations (54) and (64) for the first and third order flexural theories, respectively, one may observe that the determinant of the matrix of the frequency equation is symmetric, and its elements may be classified into four groups: A_{mn}, B_{mn} for elements on the main diagonal and C_{mn}, D_{mn} for elements off the diagonal. In terms of



(a)



(b)



(c)

FIG. 6. Dispersion curves for the fourth order extensional theory.

A_{mn}, B_{mn}, C_{mn} and D_{mn}, the dispersion relation of an Nth order flexural theory can be written as

$$
\begin{array}{cc|ccccccccc|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{9}{c}{n} \\
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & 0 & 1 & 2 & 3 & 4 & 5 & \cdots & \multicolumn{1}{c}{N} \\
 & 0 & B_{00} & D_{01} & 0 & D_{03} & 0 & D_{05} & \cdots & D_{0N} \\
 & 1 & & A_{11} & C_{12} & 0 & C_{14} & 0 & \cdots & 0 \\
 & 2 & & & B_{22} & D_{23} & 0 & D_{25} & \cdots & D_{2N} \\
m & 3 & & & & A_{33} & C_{34} & 0 & \cdots & 0 \\
 & 4 & & & & & B_{44} & D_{45} & \cdots & D_{4N} \\
 & 5 & & & & & & A_{55} & \cdots & 0 \\
 & \vdots & & & & & & & \cdots & \\
 & N & & & & & & & & A_{NN}
\end{array} = 0. \qquad (70)
$$

Similarly, by inspecting (59) and (69) of the second and fourth order extensional theories, respectively, one may write the dispersion relation of an Nth order extensional theory as

$$
\begin{array}{cc|ccccccccc|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{9}{c}{n} \\
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & 0 & 1 & 2 & 3 & 4 & 5 & \cdots & \multicolumn{1}{c}{N} \\
 & 0 & A_{00} & C_{01} & 0 & C_{03} & 0 & C_{05} & \cdots & 0 \\
 & 1 & & B_{11} & D_{12} & 0 & D_{14} & 0 & \cdots & D_{1N} \\
 & 2 & & & A_{22} & C_{23} & 0 & C_{25} & \cdots & 0 \\
m & 3 & & & & B_{33} & D_{34} & 0 & \cdots & D_{3N} \\
 & 4 & & & & & A_{44} & C_{45} & \cdots & 0 \\
 & 5 & & & & & & B_{55} & \cdots & D_{3N} \\
 & \vdots & & & & & & & \cdots & \\
 & N & & & & & & & & A_{NN}
\end{array} = 0. \qquad (71)
$$

In (69) and (70), only half the off-diagonal elements are shown due to their symmetry property, and the non-zero elements are

$$
\begin{aligned}
A_{mn} &= k^2z^2+m^2-\Omega^2, \qquad (m = n) \\
B_{mn} &= m^2k^2+z^2-\Omega^2, \qquad (m = n) \\
C_{mn} &= \mp\frac{4}{\pi(n^2-m^2)}[n^2(k^2-2)+m^2]z, \qquad (m \neq n,\ m+n = \text{odd}) \\
D_{mn} &= \pm\frac{4}{\pi(n^2-m^2)}[m^2(k^2-2)+n^2]z, \qquad (m \neq n,\ m+n = \text{odd})
\end{aligned} \qquad (72)
$$

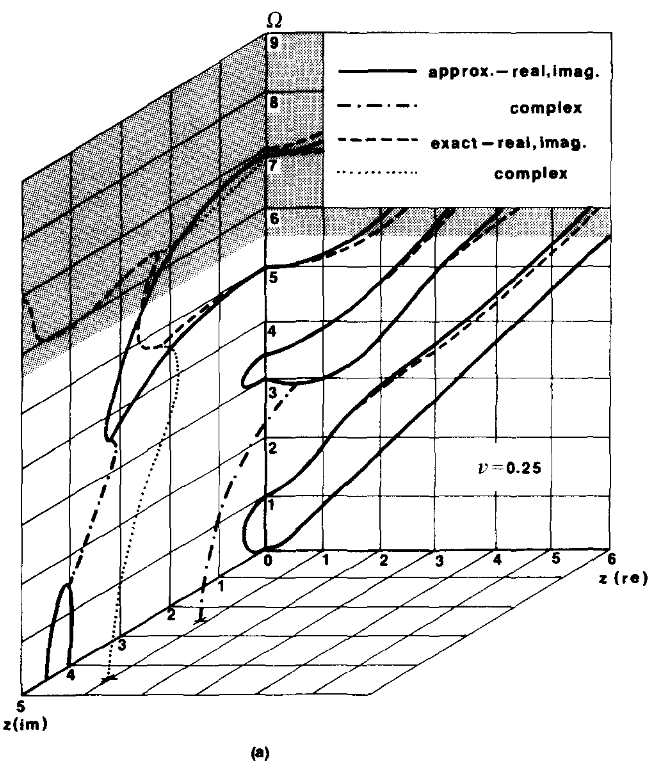
where $0 \leq m, n \leq N$ for any integer $N > 0$ and in the last two equations of (72) the upper sign is applied to the flexural theory and the lower sign to the extensional theory. By (72) one may generate elements of the determinant of either (70) for flexural theory or (71) for extensional theory to any order N, except B_{00}, B_{11}, C_{01} and D_{01}, in which α_1 and α_2 are introduced as follows

$$B_{00} = z^2 - \alpha_2^{-1}\Omega^2 \qquad B_{11} = k^2 + \alpha_2 z^2 - \Omega^2$$
$$C_{01} = \alpha_1 \frac{4}{\pi}(k^2 - 2)z, \qquad D_{01} = \alpha_1 \frac{4}{\pi} z. \tag{73}$$

Dispersion relations for the fifth order flexural theory and the sixth order extensional theory are generated by this method, and they verify the results obtained by the usual process. The dispersion curves are computed and compared with the "exact" results as shown in Figs. 7(a)–(c) for the fifth order and in Figs. 8(a)–(c) for the sixth order theory. It may be seen that all frequency branches for real, imaginary and complex wave numbers are reproduced and are in good agreement with the exact ones except the highest branch for which the complex conjugate roots, for certain values of ν, are replaced by two distinct imaginary roots. Since in these cases the magnitude of the real part is small compared to that of the imaginary part, this difference means that the mode shape of a slow oscillatory variation enveloped by a hyperbolic function is approximated by a hyperbolic function.

8. CONCLUSIONS

By an expansion in series of simple thickness-modes, two-dimensional equations of successively higher orders of approximations are derived in a systematic manner. The



(a)

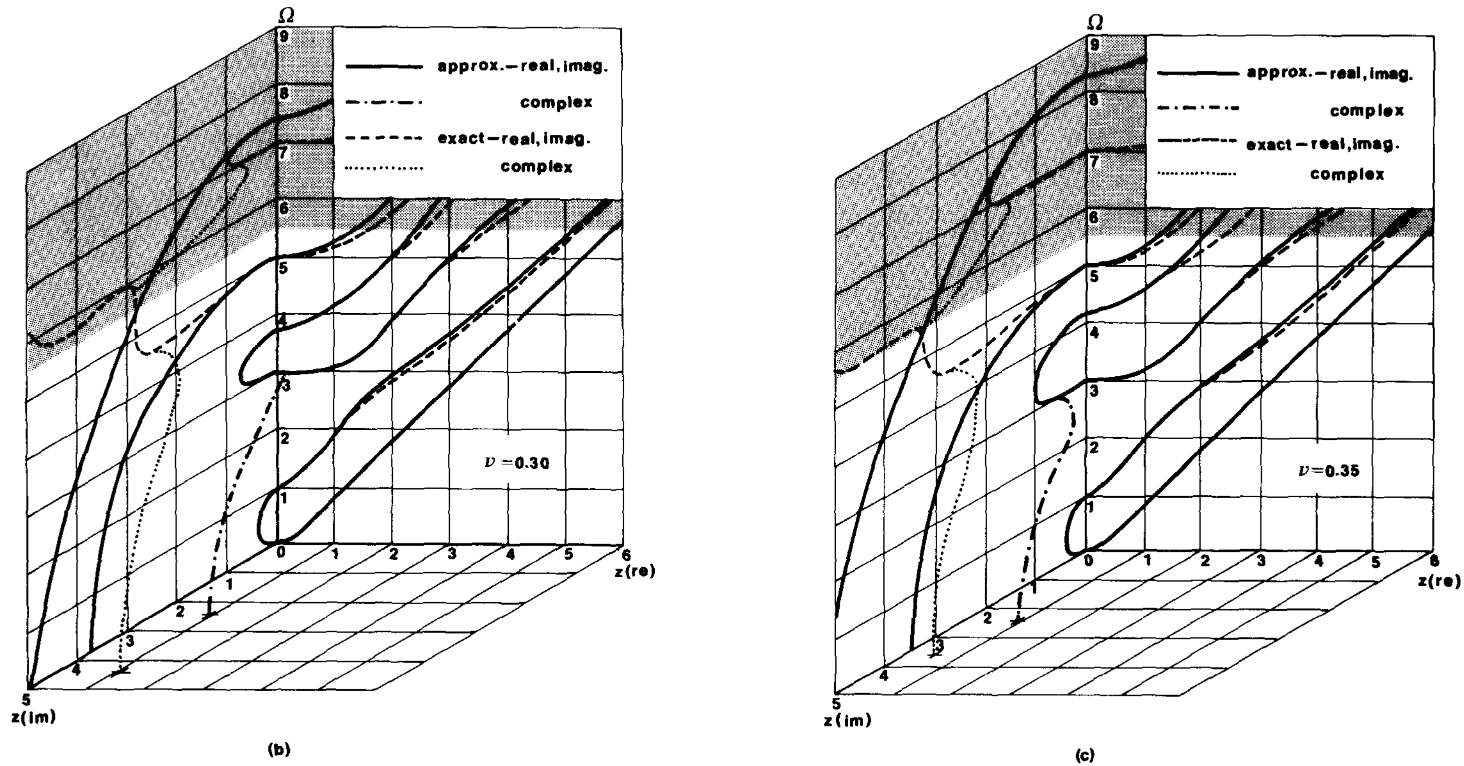
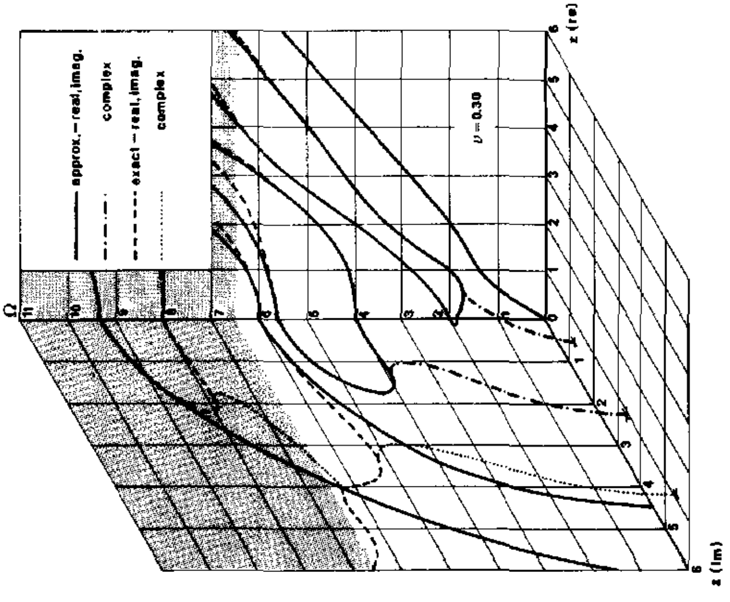
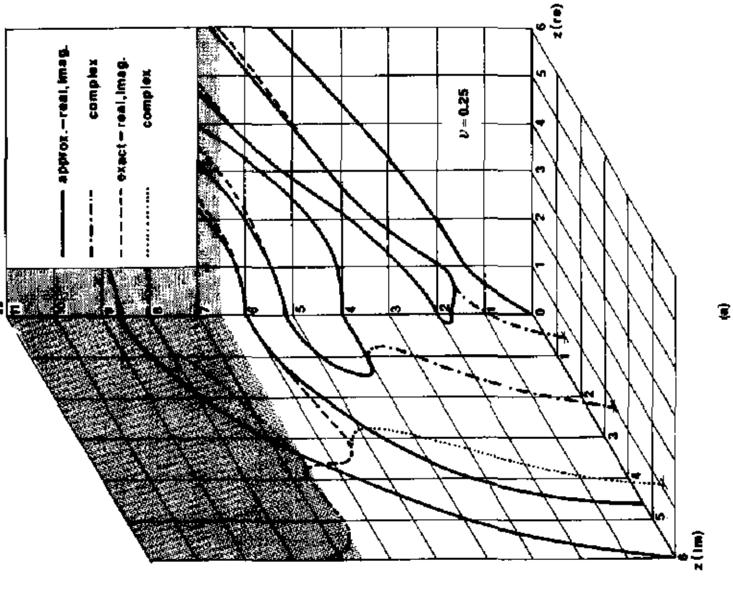


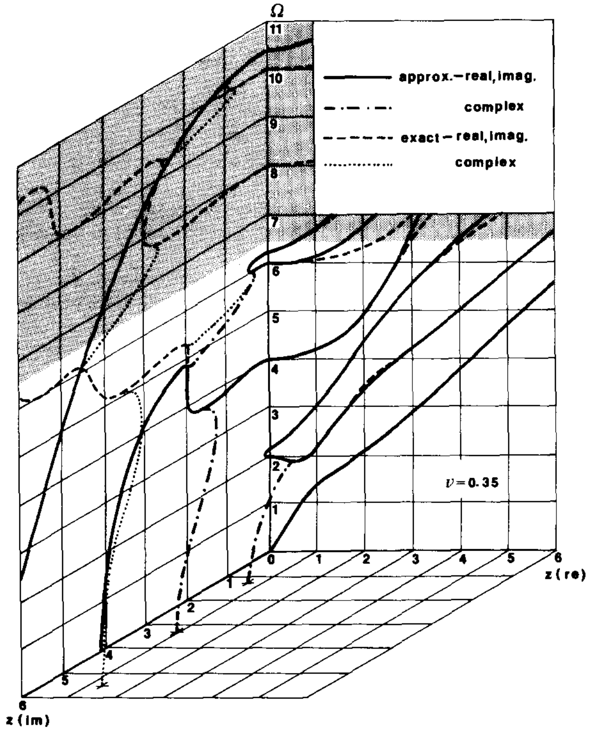
FIG. 7. Dispersion curves for the fifth order flexural theory.


approx.—real, imag.
complex
exact—real, imag.
complex
ν = 0.30
Ω
z (re)
z (Im)

(b)


approx.—real, imag.
complex
exact—real, imag.
complex
ν = 0.25
Ω
z (re)
z (Im)

(a)



(c)

FIG. 8. Dispersion curves for the sixth order extensional theory.

close agreement of the dispersion relations for both flexural and extensional vibrations with the results from the three-dimensional theory of elasticity indicates that the applicable range of frequencies for an Nth order theory can be set at $\Omega \leq N+\frac{1}{2}$. As for SH waves or face-shear vibrations in an infinite plate, the approximate equations always yield the exact dispersion relations.

Theorems of uniqueness and orthogonality can be established for each order of approximation in a similar manner to that for the case of an infinite set of equations as given in Section 3, if in addition to the usual requirements $3\lambda+2\mu > 0$, $\mu > 0$, one requires that $\alpha_2 > 0$.

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Абстракт—Из трехмерной теории упругости выводятся двухмерные уравнения постепнно высших порядков приближения для упругих, изотропных пластинок, путём разложения в ряды выражений простых видов волн сдвиговою типа для бесконечной пластинки. Для каждой степени приближения, от нулевого до четвертого, определяются плотности кинетической знергии и знергии деформации, зависимости для поля напряжений и деформаций, а также уравнения движения в перемешениях для колебаний изгиба и удлинения.

Для бесконечной пластинки следуются полробно кривые дисперсий, как для действитблвных, так и для мнимых чисел волн и сравниваются с уравнением частоты Релея–Ламба втрехмерной теорий.